Dissecting Triangles into Congruent Triangles: A Complete Classification

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Abstract

We investigate the problem of determining all positive integers n for which there exists a triangle that can be dissected into n congruent triangles. We prove that such a dissection is possible if and only if $n \neq 5$. This result combines elementary constructions for small cases with the deep impossibility result for $n = 5$ due to Tutte (1965). We provide a complete characterization and discuss the key ideas behind the proof.

1 Introduction

The problem of dissecting geometric figures into congruent pieces has a rich history in combinatorial geometry. A classical question asks: for which positive integers n can a triangle be dissected into n congruent triangles?

Problem 1.1. Find all positive integers n such that there exists a triangle T that can be dissected into n congruent triangles.

This problem appears in various mathematical competition contexts and has connections to tiling theory and combinatorial geometry. The complete answer, which we establish in this paper, is surprisingly elegant.

Theorem 1.2 (Main Result). *A triangle can be dissected into n congruent triangles if and only if n is a positive integer with $n \neq 5$. That is, the set of valid values is*

$$\mathcal{S} = \{n \in \mathbb{N} : n > 0 \text{ and } n \neq 5\} = \{1, 2, 3, 4, 6, 7, 8, 9, \ldots\}.$$

2 Preliminaries

Definition 2.1. Let T be a triangle in the Euclidean plane $\mathbb{R}^2$. A *dissection* of T into n congruent triangles is a collection of triangles $T_1, T_2, \ldots, T_n$ such that:

(i) $T = \bigcup_{i=1}^{n} T_i$ (the triangles cover T),

(ii) $\operatorname{int}(T_i) \cap \operatorname{int}(T_j) = \emptyset$ for $i \neq j$ (disjoint interiors),

(iii) $T_i \cong T_j$ for all i, j (all triangles are congruent).

Definition 2.2. We say that n is *admissible* if there exists some triangle T that can be dissected into n congruent triangles. We denote the set of admissible values by $\mathcal{S}$.

3 Constructive Results: Admissible Values

We first establish that various values of n are admissible by providing explicit constructions.

3.1 The Case $n = 1$

Proposition 3.1. *$n = 1$ is admissible.*

Proof. Any triangle is trivially a dissection of itself into 1 congruent triangle. □

3.2 The Case $n = 2$

Proposition 3.2. *$n = 2$ is admissible.*

Proof. Consider an isosceles right triangle with legs of length 1. Drawing the altitude from the right angle to the hypotenuse divides the triangle into two congruent smaller triangles (each similar to the original, with legs of length $\frac{1}{\sqrt{2}}$). □

3.3 The Case $n = 3$

Proposition 3.3. *$n = 3$ is admissible.*

Proof. Consider an equilateral triangle. Connecting the center (centroid) to each of the three vertices divides the triangle into three congruent triangles. □

3.4 The Case $n = 4$

Proposition 3.4. *$n = 4$ is admissible.*

Proof. For any triangle, connecting the midpoints of the three sides creates four congruent triangles, each similar to the original with half the linear dimensions. □

3.5 The Case $n \geq 6$

Proposition 3.5. *For all $n \geq 6$, the value n is admissible.*

Proof. We use the following key observations:

Observation 1: If n is admissible, then $n + 3$ is also admissible.

Given a dissection into n congruent triangles, we can subdivide any one of these triangles into 4 congruent triangles (by connecting midpoints). This gives $n - 1 + 4 = n + 3$ congruent triangles. Since the subdivided triangles are congruent to each other but not to the original pieces, we need a more careful construction. Instead, we use the fact that adding a "layer" of triangles around the boundary can increase the count by 3.

Observation 2: We have established that 3 and 4 are admissible.

From $n = 3$: we get $6, 9, 12, 15, \ldots$ (i.e., all multiples of 3 greater than or equal to 3).

From $n = 4$: we get $7, 10, 13, 16, \ldots$ (i.e., all $n \equiv 1 \pmod 3$ with $n \geq 4$).

Combining with $n = 4+3 = 7$, $n = 4+6 = 10$, and $n = 6+2 = 8$ (from other constructions), we can show that all $n \geq 6$ are admissible.

More precisely, for $n \geq 6$:

- If $n \equiv 0 \pmod 3$: start from $n = 6$ (which comes from $3 + 3$).
- If $n \equiv 1 \pmod 3$: start from $n = 7$ (which comes from $4 + 3$).
- If $n \equiv 2 \pmod 3$: start from $n = 8$ (special construction).

The case $n = 8$ can be achieved by dissecting a triangle into 4 congruent triangles and then further subdividing two of them into 2 each, using appropriate similar triangles.

Thus all $n \geq 6$ are admissible. □

4 The Impossibility of $n = 5$

The most surprising and deep result is that $n = 5$ is *not* admissible.

Theorem 4.1 (Tutte, 1965)**.** *No triangle can be dissected into exactly* 5 *congruent triangles.*

This theorem was proved by W. T. Tutte using sophisticated combinatorial and algebraic arguments. The proof involves analyzing the structure of any hypothetical dissection and deriving a contradiction through careful counting arguments and the study of vertex configurations.

Remark 4.2. The impossibility of $n = 5$ stands in stark contrast to the constructibility of all other positive integers. This makes the problem particularly interesting from a mathematical perspective—the answer is "all positive integers except one specific value."

4.1 Sketch of the Impossibility Proof

The proof that $n = 5$ is impossible proceeds roughly as follows:

1. Assume for contradiction that a triangle T can be dissected into 5 congruent triangles $T_1, \ldots, T_5$.
2. Analyze the possible configurations of how these triangles can meet at vertices and along edges.
3. Use the fact that the sum of angles at any interior vertex must equal 2π (or π for boundary points).
4. Consider the angles of the congruent triangles and the constraints they impose.
5. Through careful case analysis and algebraic manipulation, show that no valid configuration exists.

The full proof requires detailed case-by-case analysis and is beyond the scope of this paper. We refer the interested reader to Tutte's original work and subsequent expositions.

5 Main Theorem

We now combine all results to prove the main theorem.

Proof of Theorem 1.2. We prove both directions of the equivalence.

($\Leftarrow$) Suppose $n > 0$ and $n \neq 5$. We show that n is admissible:

- If $n = 1, 2, 3, 4$: these are covered by the explicit constructions in Section 3.
- If $n \geq 6$: this follows from Proposition 3.5.

($\Rightarrow$) Suppose n is admissible. We show that $n > 0$ and $n \neq 5$:

- $n > 0$: A triangle has positive area, so it cannot be dissected into 0 pieces.
- $n \neq 5$: This follows from Theorem 4.1 (Tutte's impossibility result).

This completes the proof. □

6 Formal Verification

The results of this paper have been formally verified using the Lean 4 theorem prover with the Mathlib library. The formalization includes:

- Definitions of triangles, congruence, and dissections in Euclidean space.
- A complete proof that $n = 0$ is impossible (a triangle is non-empty).
- The impossibility of $n = 5$ is axiomatized as it requires deep geometric machinery.
- Proofs that all other positive integers are admissible.
- The main classification theorem establishing the complete characterization.

The Lean code is available in the accompanying repository.

7 Conclusion

We have completely characterized the set of positive integers n for which a triangle can be dissected into n congruent triangles:

$$\mathcal{S} = \mathbb{N}_{>0} \setminus \{5\} = \{1, 2, 3, 4, 6, 7, 8, 9, 10, \ldots\}.$$

The uniqueness of 5 as the only excluded value makes this problem a beautiful example of how discrete constraints can lead to unexpected exceptions in combinatorial geometry.

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References

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