On Distinct Integers with Identical Prime Factor Sets for Three Consecutive Shifts

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Abstract

We prove the existence of two distinct integers x and y such that the pairs (x, y), $(x + 1, y + 1)$, and $(x + 2, y + 2)$ each share the same set of prime factors. We provide an explicit construction using $x = 2$ and $y = -4$, and verify this result using the Lean 4 theorem prover with Mathlib.

1 Introduction

The study of prime factorization and its behavior under arithmetic operations is a fundamental topic in number theory. A natural question arises: can we find distinct integers whose prime factor sets remain identical under consecutive shifts?

More precisely, we investigate whether there exist distinct integers x and y such that:

(i) x and y have the same set of prime factors,

(ii) $x + 1$ and $y + 1$ have the same set of prime factors, and

(iii) $x + 2$ and $y + 2$ have the same set of prime factors.

In this paper, we answer this question affirmatively by providing an explicit example and a formal verification in Lean 4.

2 Preliminaries

Definition 1 (Prime Factor Set)**.** For a nonzero integer $n \in \mathbb{Z}$, we define the *prime factor set* of n, denoted $\mathrm{rad}(n)$, as the set of all prime numbers that divide $|n|$:

$$\mathrm{rad}(n) = \{p \in \mathbb{P} : p \mid |n|\}$$

where $\mathbb{P}$ denotes the set of prime numbers.

Definition 2 (Same Prime Factors)**.** Two nonzero integers $a, b \in \mathbb{Z}$ are said to have the *same prime factors* if $\mathrm{rad}(a) = \mathrm{rad}(b)$.

Example 3. The integers $12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$ have the same prime factors since $\mathrm{rad}(12) = \mathrm{rad}(18) = \{2, 3\}$.

3 Main Result

Theorem 4. *There exist distinct integers $x, y \in \mathbb{Z}$ such that:*

(i) $\mathrm{rad}(x) = \mathrm{rad}(y)$*,*

(ii) $\mathrm{rad}(x+1) = \mathrm{rad}(y+1)$*, and*

(iii) $\mathrm{rad}(x+2) = \mathrm{rad}(y+2)$*.*

Proof. We claim that $x = 2$ and $y = -4$ satisfy all the required conditions.

First, we verify that $x \neq y$: clearly $2 \neq -4$.

Condition (i): We compute the prime factor sets of $x = 2$ and $y = -4$:

$$|x| = |2| = 2 = 2^1, \quad \text{so } \mathrm{rad}(x) = \{2\},$$
$$|y| = |-4| = 4 = 2^2, \quad \text{so } \mathrm{rad}(y) = \{2\}.$$

Thus $\mathrm{rad}(x) = \mathrm{rad}(y) = \{2\}$.

Condition (ii): We compute the prime factor sets of $x + 1 = 3$ and $y + 1 = -3$:

$$|x+1| = |3| = 3 = 3^1, \quad \text{so } \mathrm{rad}(x+1) = \{3\},$$
$$|y+1| = |-3| = 3 = 3^1, \quad \text{so } \mathrm{rad}(y+1) = \{3\}.$$

Thus $\mathrm{rad}(x+1) = \mathrm{rad}(y+1) = \{3\}$.

Condition (iii): We compute the prime factor sets of $x + 2 = 4$ and $y + 2 = -2$:

$$|x+2| = |4| = 4 = 2^2, \quad \text{so } \mathrm{rad}(x+2) = \{2\},$$
$$|y+2| = |-2| = 2 = 2^1, \quad \text{so } \mathrm{rad}(y+2) = \{2\}.$$

Thus $\mathrm{rad}(x+2) = \mathrm{rad}(y+2) = \{2\}$.

All three conditions are satisfied, completing the proof. □

4 Summary of the Construction

The following table summarizes the verification:

Pair	Values	Absolute Values	Prime Factors	Match
(x, y)	$(2, -4)$	$(2, 4)$	$\{2\}, \{2\}$	✓
$(x+1, y+1)$	$(3, -3)$	$(3, 3)$	$\{3\}, \{3\}$	✓
$(x+2, y+2)$	$(4, -2)$	$(4, 2)$	$\{2\}, \{2\}$	✓

5 Formal Verification in Lean 4

The theorem has been formally verified using the Lean 4 theorem prover with the Mathlib library. The key definitions and theorem statement are as follows:

```
def SamePrimeFactors (a b : Int) : Prop :=
  a.natAbs.primeFactors = b.natAbs.primeFactors

theorem exists_distinct_integers_with_same_prime_factors_consecutive_3 :
  exists x y : Int, x != y /\
    SamePrimeFactors x y /\
    SamePrimeFactors (x + 1) (y + 1) /\
    SamePrimeFactors (x + 2) (y + 2) := by
  use 2, -4
  simp +decide [SamePrimeFactors]
  native_decide
```

Note: In the actual Lean 4 code, Unicode symbols are used: `Int` is written as $\mathbb{Z}$, `exists` as $\exists$, `!=` as $\neq$, and `/\` as $\wedge$.

The proof is completed automatically by Lean's `native_decide` tactic, which performs a computational verification of the prime factor equality.

6 Remarks

Remark 5. The solution $(x, y) = (2, -4)$ exploits the fact that negation preserves prime factors (since $\text{rad}(n) = \text{rad}(-n)$), and the specific arithmetic relationships:

- 2 and -4 differ by 6, and both are powers of 2.
- 3 and -3 are negatives of each other.
- 4 and -2 are both powers of 2 (with opposite signs considered).

Remark 6. If we restrict to positive integers, the problem becomes more challenging. A computational search suggests that finding positive solutions requires significantly larger values. The question of whether positive integer solutions exist, and if so, finding the minimal such pair, remains an interesting open problem.

7 Conclusion

We have proven the existence of distinct integers x and y such that the pairs (x, y), $(x+1, y+1)$, and $(x+2, y+2)$ each share identical prime factor sets. The explicit solution $(x, y) = (2, -4)$ was verified both by direct computation and through formal verification in the Lean 4 theorem prover.

References

[1] The Mathlib Community. *Mathlib4: The math library for Lean 4*. `https://github.com/leanprover-community/mathlib4`, 2024.

[2] Leonardo de Moura and Sebastian Ullrich. *The Lean 4 Theorem Prover and Programming Language*. In *CADE-28*, 2021.