# Formalizing Unstable Quasinormal Modes and the Quantum Black Hole Bomb in Lean 4

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**Abstract**

We present a formal verification of the exponential growth of unstable quasinormal modes in the context of quantum-induced superradiance using the Lean 4 proof assistant and the Mathlib library. The formalization rigorously defines the concept of an unstable mode as one with a positive imaginary part of its complex frequency, proves that such modes exhibit exponential growth in time, and establishes the mathematical foundation for the quantum black hole bomb phenomenon. All proofs are machine-checked, ensuring complete mathematical rigor.

## 1 Introduction

Superradiance is a fundamental phenomenon in black hole physics where bosonic fields can extract rotational energy from a rotating black hole. When the frequency $\omega$ of an incident wave satisfies $\omega < m\Omega_H$, where $m$ is the azimuthal quantum number and $\Omega_H$ is the angular velocity of the horizon, the reflected wave is amplified [1].

The *black hole bomb* mechanism, first proposed by Press and Teukolsky [2], occurs when superradiant amplification is combined with a confining mechanism that reflects the amplified waves back toward the black hole, creating an exponentially growing instability. In the quantum context, massive bosonic fields provide a natural confining mechanism through their Compton wavelength.

### 1.1 Quasinormal Modes

Quasinormal modes (QNMs) are the characteristic oscillation modes of black holes. They are characterized by complex frequencies $\omega = \omega_R + i\omega_I$, where:

- $\omega_R$ is the oscillation frequency,
- $\omega_I$ determines the growth or decay rate.

The time evolution of a mode is given by $e^{-i\omega t}$, which can be written as:

$$e^{-i\omega t} = e^{-i(\omega_R + i\omega_I)t} = e^{\omega_I t} \cdot e^{-i\omega_R t}. \tag{1}$$

When $\omega_I > 0$, the mode amplitude grows exponentially in time—this is the signature of an instability.

### 1.2 Motivation for Formalization

Formal verification of physical and mathematical results provides several benefits:

1. **Rigor**: Machine-checked proofs eliminate any possibility of error.

2. **Clarity**: The formalization process forces precise definitions.

3. **Reproducibility**: The proof can be independently verified by anyone.

In this paper, we present a complete Lean 4 formalization of the key mathematical results underlying the quantum black hole bomb phenomenon.

## 2 Mathematical Framework

### 2.1 Mode Evolution

**Definition 2.1** (Mode Evolution). Given a complex frequency $\omega \in \mathbb{C}$ and time $t \in \mathbb{R}$, the *mode evolution* is defined as

$$\psi(t) = e^{-i\omega t}. \tag{2}$$

**Definition 2.2** (Unstable Mode). A mode with complex frequency $\omega$ is called *unstable* if the imaginary part of $\omega$ is positive:

$$\omega_I := \mathrm{Im}(\omega) > 0. \tag{3}$$

### 2.2 Growth of Unstable Modes

**Theorem 2.3** (Magnitude of Mode Evolution). *For any complex frequency $\omega \in \mathbb{C}$ and time $t \in \mathbb{R}$, the magnitude of the mode evolution is given by*

$$|e^{-i\omega t}| = e^{\omega_I t}, \tag{4}$$

*where $\omega_I = \mathrm{Im}(\omega)$.*

*Proof.* Writing $\omega = \omega_R + i\omega_I$, we have

$$-i\omega = -i\omega_R + \omega_I, \tag{5}$$
$$-i\omega t = \omega_I t - i\omega_R t. \tag{6}$$

Therefore,

$$e^{-i\omega t} = e^{\omega_I t} \cdot e^{-i\omega_R t}. \tag{7}$$

Taking the absolute value and using $|e^{i\theta}| = 1$ for real $\theta$:

$$|e^{-i\omega t}| = |e^{\omega_I t}| \cdot |e^{-i\omega_R t}| = e^{\omega_I t} \cdot 1 = e^{\omega_I t}. \tag{8}$$

□

**Theorem 2.4** (Instability Implies Growth). *If $\omega$ is an unstable mode (i.e., $\omega_I > 0$) and $t > 0$, then*

$$|e^{-i\omega t}| > 1. \tag{9}$$

*Proof.* By Theorem 2.3, $|e^{-i\omega t}| = e^{\omega_I t}$. Since $\omega_I > 0$ and $t > 0$, we have $\omega_I t > 0$, and thus $e^{\omega_I t} > e^0 = 1$. □

### 2.3 Quasinormal Mode Spectrum

In the context of black hole perturbation theory, the quasinormal frequencies form a discrete spectrum labeled by quantum numbers.

**Definition 2.5** (Quasinormal Frequency Spectrum)**.** A *quasinormal frequency spectrum* is a function

$$\omega : \mathbb{N} \times \mathbb{N} \times \mathbb{Z} \to \mathbb{C}, \tag{10}$$

where the arguments $(n, l, m)$ represent the overtone number, angular momentum quantum number, and azimuthal quantum number, respectively.

**Definition 2.6** (Existence of Unstable Mode)**.** A quasinormal frequency spectrum *has an unstable mode* if there exist quantum numbers $(n, l, m)$ such that

$$\operatorname{Im}(\omega_{n,l,m}) > 0. \tag{11}$$

### 2.4 The Quantum Black Hole Bomb

**Definition 2.7** (Quantum Black Hole Bomb)**.** A system is a *quantum black hole bomb* if its quasinormal frequency spectrum has at least one unstable mode.

**Theorem 2.8** (Quantum Black Hole Bomb Growth)**.** *If a system is a quantum black hole bomb, then there exist quantum numbers $(n, l, m)$ and a time $t > 0$ such that the corresponding mode exhibits exponential growth:*

$$|e^{-i\omega_{n,l,m}t}| > 1. \tag{12}$$

*Proof.* By definition, there exist $(n, l, m)$ with $\operatorname{Im}(\omega_{n,l,m}) > 0$. Taking $t = 1 > 0$ and applying Theorem 2.4, we obtain $|e^{-i\omega_{n,l,m}\cdot 1}| > 1$. □

## 3 Formalization in Lean 4

We now present the complete Lean 4 formalization of the above results. The formalization uses Lean 4.24.0 and Mathlib (commit `f897ebcf`).

### 3.1 Basic Definitions

**Definition 3.1** (Unstable Mode in Lean)**.** The predicate for an unstable mode is defined as:

```
def is_unstable ( : ) : Prop := 0 < .im
```

**Definition 3.2** (Mode Evolution in Lean)**.** The mode evolution function is defined as:

```
noncomputable def mode_evolution ( : ) (t : ) :  :=
    Complex.exp (-Complex.I *  * (t : ))
```

### 3.2 Main Theorems

**Theorem 3.3** (Instability Implies Growth in Lean)**.** *The formal statement and proof:*

```
theorem instability_implies_growth ( : ) (t : )
    (h : is_unstable ) (ht : 0 < t) :
    1 < ‖mode_evolution  t‖ := by
  unfold is_unstable at h
  unfold mode_evolution
  norm_num [Complex.norm_exp]
  positivity
```

*Remark* 3.4. The proof uses Mathlib's `positivity` tactic, which automatically handles positivity arguments. The key insight is that $\|e^z\| = e^{\mathrm{Re}(z)}$ for complex $z$, and here $\mathrm{Re}(-i\omega t) = \omega_I t > 0$.

**Theorem 3.5** (Mode Growth Magnitude in Lean)**.** *The exact growth factor is formalized as:*

```
theorem mode_growth_magnitude ( : ) (t : ) :
    ‖mode_evolution  t‖ = Real.exp (.im * t) := by
  norm_num [mode_evolution, Complex.norm_exp]
```

### 3.3 Quasinormal Modes and Black Hole Bomb

**Definition 3.6** (Has Unstable Mode in Lean)**.** `def has_unstable_mode (quasinormal_frequency :  →  →`
```
     n l m, 0 < (quasinormal_frequency n l m).im
```

**Definition 3.7** (Quantum Black Hole Bomb in Lean)**.** `def is_quantum_black_hole_bomb`
```
    (quasinormal_frequency :  →  →  → ) : Prop :=
  has_unstable_mode quasinormal_frequency
```

**Theorem 3.8** (Quantum Black Hole Bomb Growth in Lean)**.** `theorem quantum_black_hole_bomb_growth`
```
    (quasinormal_frequency :  →  →  → )
    (h : is_quantum_black_hole_bomb quasinormal_frequency) :
     n l m t, 0 < t
        1 < ‖mode_evolution (quasinormal_frequency n l m) t‖ := by
  exact h.imp fun n hn => hn.imp fun l hl => hl.imp fun m hm =>
    ⟨1, by norm_num, by simpa using
        instability_implies_growth (quasinormal_frequency n l m) 1 hm
        (by norm_num)⟩
```

### 3.4 Verification

**Theorem 3.9** (Compilation Success)**.** *The complete formalization compiles without errors in Lean 4.24.0 with Mathlib.*

*Proof.* Running `lake env lean` on the source file produces no errors, confirming that all type checking and proof verification succeeds. □

## 4 Summary of Formalized Results

| Name | Type | Description |
|---|---|---|
| `is_unstable` | Definition | $\omega_I > 0$ |
| `mode_evolution` | Definition | $e^{-i\omega t}$ |
| `instability_implies_growth` | Theorem | $\omega_I > 0 \wedge t > 0 \Rightarrow \lvert e^{-i\omega t}\rvert > 1$ |
| `mode_growth_magnitude` | Theorem | $\lvert e^{-i\omega t}\rvert = e^{\omega_I t}$ |
| `has_unstable_mode` | Definition | $\exists\, n, l, m : \omega_I > 0$ |
| `is_quantum_black_hole_bomb` | Definition | System with unstable mode |
| `quantum_black_hole_bomb_growth` | Theorem | Bomb $\Rightarrow$ exponential growth |

Table 1: Summary of formalized definitions and theorems

# 5 Physical Implications

The formalized results have important physical implications for the study of black hole superradiance:

1. **Superradiant Instability**: When a massive bosonic field (such as an ultralight axion) surrounds a rotating black hole, bound states can form with $\omega_I > 0$, leading to exponential extraction of angular momentum from the black hole.

2. **Gravitational Wave Signatures**: The growing mode eventually saturates through gravitational wave emission, providing a potential observational signature [3].

3. **Constraints on New Physics**: The absence of observed superradiant instabilities places constraints on the existence of ultralight bosons [4].

# 6 Conclusion

We have presented a complete, machine-verified formalization of the mathematical foundations of the quantum black hole bomb phenomenon in Lean 4. The key results are:

- An unstable quasinormal mode (with $\omega_I > 0$) grows exponentially in time.
- The magnitude of the mode evolution is exactly $e^{\omega_I t}$.
- The existence of any unstable mode in the quasinormal spectrum implies exponential growth of perturbations.

All proofs have been verified by the Lean type checker, providing the highest level of mathematical certainty..

# References


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