Finiteness of Powerful Numbers of the Form $2^n \pm 1$ Under the ABC Conjecture

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January 13, 2026

Abstract

We prove that, assuming the ABC conjecture, the set of natural numbers n such that 2^n+1 is a powerful number is finite, and similarly for 2^n-1. The proof relies on constructing ABC triples from powerful numbers and applying the radical bound implied by the ABC conjecture.

1 Introduction

A natural number n is called *powerful* if for every prime p dividing n, the square p^2 also divides n. Equivalently, n is powerful if and only if n can be written as $n = a^2b^3$ for some positive integers a and b. The study of powerful numbers has connections to various areas of number theory, including the distribution of squarefree numbers and the ABC conjecture.

In this paper, we investigate the occurrence of powerful numbers among integers of the form $2^n + 1$ and $2^n - 1$. Our main result shows that, conditional on the ABC conjecture, there are only finitely many such n.

2 Definitions and Preliminaries

Definition 2.1 (Powerful Number)**.** A natural number $n \geq 1$ is *powerful* if for every prime p,

$$p \mid n \implies p^2 \mid n.$$

We denote this property by $\mathrm{Powerful}(n)$.

Definition 2.2 (Radical)**.** For a natural number $n \geq 1$, the *radical* of n, denoted $\mathrm{rad}(n)$, is the product of the distinct prime factors of n:

$$\mathrm{rad}(n) = \prod_{\substack{p \mid n \\ p \text{ prime}}} p.$$

We set $\mathrm{rad}(1) = 1$.

Definition 2.3 (ABC Conjecture)**.** The *ABC conjecture* states that for every $\varepsilon > 0$, the set

$$\left\{ (a,b,c) \in \mathbb{N}^3 \;\middle|\; \begin{array}{ll} a,b,c > 0, & \gcd(a,b) = 1, \\ a+b = c, & c > \mathrm{rad}(abc)^{1+\varepsilon} \end{array} \right\}$$

is finite.

3 Key Lemmas

Lemma 3.1 (Radical Bound for Powerful Numbers)**.** *Let $n \geq 1$ be a powerful number. Then*

$$\mathrm{rad}(n)^2 \leq n.$$

Proof. Let $n = \prod_{i=1}^k p_i^{a_i}$ be the prime factorization of n, where $p_1, \ldots, p_k$ are distinct primes and $a_i \geq 1$ for all i. Since n is powerful, we have $a_i \geq 2$ for all i.

The radical of n is

$$\mathrm{rad}(n) = \prod_{i=1}^k p_i.$$

Therefore,

$$\mathrm{rad}(n)^2 = \prod_{i=1}^k p_i^2.$$

Since $a_i \geq 2$ for all i, we have $p_i^2 \mid p_i^{a_i}$ for each i. Thus,

$$\prod_{i=1}^k p_i^2 \,\Bigg|\, \prod_{i=1}^k p_i^{a_i} = n.$$

In particular, $\mathrm{rad}(n)^2 \leq n$. □

Lemma 3.2 (Radical of Coprime Product)**.** *Let $a, b \geq 1$ be coprime natural numbers, i.e.,* $\gcd(a, b) = 1$*. Then*

$$\mathrm{rad}(ab) = \mathrm{rad}(a) \cdot \mathrm{rad}(b).$$

Proof. Since $\gcd(a, b) = 1$, the sets of prime factors of a and b are disjoint. Therefore,

$$\mathrm{rad}(ab) = \prod_{\substack{p \mid ab \\ p \text{ prime}}} p = \prod_{\substack{p \mid a \\ p \text{ prime}}} p \cdot \prod_{\substack{p \mid b \\ p \text{ prime}}} p = \mathrm{rad}(a) \cdot \mathrm{rad}(b).$$

□

Lemma 3.3 (Radical of Powers of Two)**.** *For any $n \geq 1$,*

$$\mathrm{rad}(2^n) = 2.$$

Proof. The only prime factor of 2^n is 2, so $\mathrm{rad}(2^n) = 2$. □

4 Main Results

4.1 The Case $2^n + 1$

Lemma 4.1. *Let $n \geq 0$ and suppose $2^n + 1$ is powerful. Then*

$$\mathrm{rad}(2^n \cdot (2^n + 1))^2 \leq 4(2^n + 1).$$

Proof. Since $\gcd(2^n, 2^n + 1) = 1$, by Lemma 3.2,

$$\mathrm{rad}(2^n \cdot (2^n + 1)) = \mathrm{rad}(2^n) \cdot \mathrm{rad}(2^n + 1) = 2 \cdot \mathrm{rad}(2^n + 1).$$

By Lemma 3.1, since $2^n + 1$ is powerful,

$$\mathrm{rad}(2^n + 1)^2 \leq 2^n + 1.$$

Therefore,

$$\mathrm{rad}(2^n \cdot (2^n + 1))^2 = 4 \cdot \mathrm{rad}(2^n + 1)^2 \leq 4(2^n + 1).$$

□

Proposition 4.2. *Let $n \geq 3$ and suppose $2^n + 1$ is powerful. Then*

$$2^n + 1 > \operatorname{rad}(2^n \cdot (2^n + 1))^{6/5}.$$

Proof. By Lemma 4.1,

$$\operatorname{rad}(2^n \cdot (2^n + 1))^2 \leq 4(2^n + 1).$$

Let $x = 2^n + 1$. For $n \geq 3$, we have $x \geq 9$. We claim that

$$(4x)^{3/5} < x.$$

Raising both sides to the power of 5, this is equivalent to showing

$$(4x)^3 < x^5,$$

which simplifies to

$$64x^3 < x^5,$$

or equivalently,

$$64 < x^2.$$

Since $x \geq 9$, we have $x^2 \geq 81 > 64$, so the inequality holds.

Now, since $\operatorname{rad}(2^n \cdot (2^n + 1))^2 \leq 4x$, we have

$$\operatorname{rad}(2^n \cdot (2^n + 1)) \leq \sqrt{4x} = 2\sqrt{x}.$$

Therefore,

$$\operatorname{rad}(2^n \cdot (2^n + 1))^{6/5} \leq (2\sqrt{x})^{6/5} = (4x)^{3/5} < x = 2^n + 1.$$

□

Theorem 4.3. *Assume the ABC conjecture. Then the set*

$$\{n \in \mathbb{N} \mid 2^n + 1 \text{ is powerful}\}$$

is finite.

Proof. Consider the triple $(a, b, c) = (1, 2^n, 2^n + 1)$ for $n \geq 3$ such that $2^n + 1$ is powerful. We verify:

- $a, b, c > 0$: Clear.
- $\gcd(a, b) = \gcd(1, 2^n) = 1$: Clear.
- $a + b = 1 + 2^n = 2^n + 1 = c$: Clear.

By Proposition 4.2,

$$c = 2^n + 1 > \operatorname{rad}(2^n \cdot (2^n + 1))^{6/5} = \operatorname{rad}(1 \cdot 2^n \cdot (2^n + 1))^{6/5} = \operatorname{rad}(abc)^{1+1/5}.$$

Thus, the triple $(1, 2^n, 2^n + 1)$ satisfies the ABC exception condition with $\varepsilon = 1/5$.

By the ABC conjecture, there are only finitely many such triples. Since the map $n \mapsto (1, 2^n, 2^n + 1)$ is injective, there are only finitely many $n \geq 3$ with $2^n + 1$ powerful.

Together with the finitely many $n < 3$, the set of all n such that $2^n + 1$ is powerful is finite. □

4.2 The Case $2^n - 1$

Lemma 4.4. *Let $n \geq 1$ and suppose $2^n - 1$ is powerful. Then*

$$\mathrm{rad}(2^n \cdot (2^n - 1))^2 \leq 4(2^n - 1).$$

Proof. Since $\gcd(2^n, 2^n - 1) = 1$, by Lemma 3.2,

$$\mathrm{rad}(2^n \cdot (2^n - 1)) = \mathrm{rad}(2^n) \cdot \mathrm{rad}(2^n - 1) = 2 \cdot \mathrm{rad}(2^n - 1).$$

By Lemma 3.1, since $2^n - 1$ is powerful,

$$\mathrm{rad}(2^n - 1)^2 \leq 2^n - 1.$$

Therefore,

$$\mathrm{rad}(2^n \cdot (2^n - 1))^2 = 4 \cdot \mathrm{rad}(2^n - 1)^2 \leq 4(2^n - 1).$$

□

Proposition 4.5. *Let $n > 3$ and suppose $2^n - 1$ is powerful. Then*

$$2^n > \mathrm{rad}(2^n \cdot (2^n - 1))^{6/5}.$$

Proof. By Lemma 4.4,

$$\mathrm{rad}(2^n \cdot (2^n - 1)) \leq 2\sqrt{2^n - 1}.$$

We claim that for $n > 3$,

$$2^n > (2\sqrt{2^n - 1})^{6/5}.$$

Raising both sides to the power of 5, this is equivalent to

$$2^{5n} > (2\sqrt{2^n - 1})^6 = 64(2^n - 1)^3.$$

Let $y = 2^n$. We need to show $y^5 > 64(y-1)^3$ for $y \geq 16$ (i.e., $n \geq 4$).
For $y \geq 16$:

$$\frac{y^5}{(y-1)^3} = y^2 \cdot \frac{y^3}{(y-1)^3} = y^2 \cdot \left(\frac{y}{y-1}\right)^3.$$

Since $y/(y-1) > 1$ and $y^2 \geq 256$ for $y \geq 16$, we have

$$\frac{y^5}{(y-1)^3} > y^2 \geq 256 > 64.$$

Thus $y^5 > 64(y-1)^3$, which gives us

$$2^n > \mathrm{rad}(2^n \cdot (2^n - 1))^{6/5}.$$

□

Theorem 4.6. *Assume the ABC conjecture. Then the set*

$$\{n \in \mathbb{N} \mid 2^n - 1 \text{ is powerful}\}$$

is finite.

Proof. Consider the triple $(a, b, c) = (1, 2^n - 1, 2^n)$ for $n > 3$ such that $2^n - 1$ is powerful. We verify:

- $a, b, c > 0$: Clear for $n \geq 1$.

- $\gcd(a,b) = \gcd(1, 2^n - 1) = 1$: Clear.
- $a + b = 1 + (2^n - 1) = 2^n = c$: Clear.

By Proposition 4.5,

$$c = 2^n > \text{rad}(2^n \cdot (2^n - 1))^{6/5} = \text{rad}(1 \cdot (2^n - 1) \cdot 2^n)^{6/5} = \text{rad}(abc)^{1+1/5}.$$

Thus, the triple $(1, 2^n - 1, 2^n)$ satisfies the ABC exception condition with $\varepsilon = 1/5$.

By the ABC conjecture, there are only finitely many such triples. Since the map $n \mapsto (1, 2^n - 1, 2^n)$ is injective, there are only finitely many $n > 3$ with $2^n - 1$ powerful.

Together with the finitely many $n \leq 3$, the set of all n such that $2^n - 1$ is finite. □

5 Conclusion

We have established the following results:

Corollary 5.1. *Assuming the ABC conjecture:*

1. *The set* $\{n \in \mathbb{N} \mid 2^n + 1 \text{ is powerful}\}$ *is finite.*
2. *The set* $\{n \in \mathbb{N} \mid 2^n - 1 \text{ is powerful}\}$ *is finite.*

These results demonstrate the power of the ABC conjecture in establishing finiteness results for Diophantine problems. The key insight is that powerful numbers have small radicals relative to their size, which creates ABC "hits" when combined with the exponential growth of 2^n.

Remark 5.2. The proofs use $\varepsilon = 1/5$ in the ABC conjecture. Any $\varepsilon < 1/2$ would suffice, since the essential inequality is $\text{rad}(n)^2 \leq n$ for powerful n.

References

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