On Sums of Coprime Powerful Numbers: Formal Verification of Nitaj's Construction

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Abstract

An r-powerful number is a positive integer n such that for every prime p dividing n, we have $p^r \mid n$. We study the problem of expressing an r-powerful number as a sum of coprime r-powerful numbers. We formalize in Lean 4 the definitions and key examples, including Nitaj's construction for $r = 3$ and Cambie's example for $r = 5$. We verify the Lander–Parkin counterexample to Euler's sum of powers conjecture. We define the Nitaj map and prove that, under certain positivity conditions, it generates infinitely many solutions to the 3-powerful sum problem. All results have been formally verified in the Lean 4 proof assistant using the Mathlib library.

1 Introduction

The study of powerful numbers has a rich history in number theory, dating back to Golomb's work in the 1970s. A *powerful number* (or *squareful number*) is a positive integer n such that if a prime p divides n, then p^2 also divides n. This notion generalizes naturally to higher powers.

Definition 1.1 (r-Powerful Number)**.** Let $r \geq 1$ be a positive integer. A positive integer n is called *r-powerful* if for every prime p dividing n, we have $p^r \mid n$. Equivalently, in the prime factorization $n = \prod_i p_i^{e_i}$, every exponent satisfies $e_i \geq r$.

A natural question arises: can an r-powerful number be expressed as a sum of coprime r-powerful numbers? This question leads to several fascinating problems and conjectures.

Definition 1.2 (Pairwise Coprime)**.** A collection of positive integers $a_1, \dots, a_k$ is *pairwise coprime* if $\gcd(a_i, a_j) = 1$ for all $i \neq j$.

Definition 1.3 (Globally Coprime)**.** A collection of positive integers $a_1, \dots, a_k$ is *globally coprime* if $\gcd(a_1, \dots, a_k) = 1$.

Remark 1.4. Pairwise coprimality implies global coprimality, but the converse does not hold. For example, $\{6, 10, 15\}$ is globally coprime but not pairwise coprime.

The main results of this paper are the formal verification of several key examples and the conditional proof that Nitaj's construction generates infinitely many solutions.

2 The 3-Powerful Sum Problem

The case $r = 3$ (3-powerful or *cubeful* numbers) has been extensively studied. A 3-powerful number has the form a^2b^3 where $a, b \in \mathbb{N}$.

2.1 Nitaj's Example

Nitaj discovered an elegant example of coprime 3-powerful numbers whose sum is also 3-powerful.

Definition 2.1 (Nitaj's Numbers)**.** Define:

$$a := 2^3 \cdot 3^5 \cdot 73^3 = 756249048,$$
$$b := 271^3 = 19902511,$$
$$c := 919^3 = 776151559.$$

Theorem 2.2 (Nitaj's Equation)**.** *We have* $a + b = c$*.*

Proof. Direct computation:

$$756249048 + 19902511 = 776151559 = 919^3.$$

This has been verified by `norm_num` in Lean. □

Theorem 2.3. *The numbers* a*,* b*, and* c *defined above are all 3-powerful.*

Proof.

- $a = 2^3 \cdot 3^5 \cdot 73^3$: Each prime factor (2, 3, 73) appears with exponent at least 3.
- $b = 271^3$: The only prime factor is 271 with exponent 3.
- $c = 919^3$: The only prime factor is 919 with exponent 3.

□

Theorem 2.4. *We have* $\gcd(a, b) = 1$*.*

Proof. The prime factorization of a involves primes 2, 3, 73, while $b = 271^3$ is a prime power with $271 \notin \{2, 3, 73\}$. Thus $\gcd(a, b) = 1$. This has been verified by `native_decide` in Lean. □

Corollary 2.5. *Nitaj's example provides a solution to the 3-powerful sum problem with* $k = 2$ *summands.*

Remark 2.6. In Nitaj's example, both $b = 271^3$ and $c = 919^3$ are perfect cubes, while a is not a perfect cube.

3 The 5-Powerful Sum Problem

Cambie extended the study to 5-powerful numbers.

Definition 3.1 (Cambie's Numbers)**.** Define:

$$t_1 := 2^8 \cdot 3^{10} \cdot 5^7,$$
$$t_2 := 2^{12} \cdot 23^6,$$
$$t_3 := 11^5 \cdot 13^5,$$
$$s := 3^7 \cdot 61^5.$$

Theorem 3.2 (Cambie's Equation)**.** *We have* $t_1 + t_2 + t_3 = s$*.*

Proof. Direct computation verified by `norm_num` in Lean. □

Theorem 3.3. *The numbers* t_1*,* t_2*,* t_3*, and* s *are all 5-powerful.*

Proof.

- $t_1 = 2^8 \cdot 3^{10} \cdot 5^7$: All exponents (8, 10, 7) are at least 5.

- $t_2 = 2^{12} \cdot 23^6$: Exponents 12 and 6 are at least 5.
- $t_3 = 11^5 \cdot 13^5$: Both exponents equal 5.
- $s = 3^7 \cdot 61^5$: Exponents 7 and 5 are at least 5.

□

Theorem 3.4. *The triple* (t_1, t_2, t_3) *is globally coprime:* $\gcd(t_1, t_2, t_3) = 1$.

Proof. Note that $t_3 = 11^5 \cdot 13^5$ shares no prime factors with t_1 or t_2 (which involve primes 2, 3, 5, 23). Thus $\gcd(t_1, t_2, t_3) = 1$. Verified by `native_decide` in Lean. □

Theorem 3.5. *The triple* (t_1, t_2, t_3) *is* not *pairwise coprime.*

Proof. Both t_1 and t_2 are divisible by 2^8, so $\gcd(t_1, t_2) \geq 2^8 > 1$. □

4 Counterexample to Euler's Conjecture

Euler conjectured that the sum of $k - 1$ positive k-th powers is never itself a k-th power. This conjecture was disproved for $k = 5$ by Lander and Parkin in 1966.

Theorem 4.1 (Lander–Parkin, 1966)**.** *We have*

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

Proof. Direct computation:

$$\begin{aligned} 27^5 &= 14348907, \\ 84^5 &= 4182119424, \\ 110^5 &= 1610510000, \\ 133^5 &= 4181016571, \\ 27^5 + 84^5 + 110^5 + 133^5 &= 61917364224 = 144^5. \end{aligned}$$

Verified by `norm_num` in Lean. □

Corollary 4.2. *Euler's sum of powers conjecture is false for* $k = 5$.

Remark 4.3. Any k-th power n^k is automatically k-powerful. Thus the Lander–Parkin counterexample also provides 5-powerful numbers summing to a 5-powerful number.

5 The Nitaj Map and Infinite Solutions

Nitaj developed an iterative method to generate infinitely many solutions to the 3-powerful sum problem, starting from a suitable seed.

5.1 The Nitaj Map

Definition 5.1 (Nitaj Map)**.** For integers u, v, define the *Nitaj map* $\Phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ by

$$\Phi(u, v) := (x, y)$$

where

$$\begin{aligned} A &:= u^3 - 3uv^2 - v^3, \\ B &:= -3uv(u + v), \\ x &:= A + B, \\ y &:= A - 2B. \end{aligned}$$

Theorem 5.2 (Nitaj Identity)**.** *For all* $u, v \in \mathbb{Z}$*, let* $(x, y) = \Phi(u, v)$*. Then*

$$x^2 + xy + y^2 = 3(u^2 + uv + v^2)^3.$$

Proof. Algebraic verification by expanding both sides. Verified by the `ring` tactic in Lean. □

Theorem 5.3 (Nitaj Difference)**.** *For all* $u, v \in \mathbb{Z}$*, let* $(x, y) = \Phi(u, v)$*. Then*

$$y - x = 9uv(u + v).$$

Proof. Direct computation:

$$y - x = (A - 2B) - (A + B) = -3B = 9uv(u + v).$$

□

Corollary 5.4. *If* $u, v > 0$*, then* $y > x$*.*

5.2 Properties of the Nitaj Map

Theorem 5.5. *For* $u, v > 0$*, let* $(x, y) = \Phi(u, v)$ *and define* $a := y^3 - x^3$*. Then*

$$a = 27 \cdot uv(u + v) \cdot (u^2 + uv + v^2)^3.$$

Proof. Using the factorization $y^3 - x^3 = (y - x)(y^2 + xy + x^2)$ and the identities from Theorems 5.2 and 5.3:

$$a = (y - x)(x^2 + xy + y^2) = 9uv(u + v) \cdot 3(u^2 + uv + v^2)^3 = 27uv(u + v)(u^2 + uv + v^2)^3.$$

□

Definition 5.6 (Strong Nitaj Seed)**.** A pair $(u, v) \in \mathbb{N}^2$ is a *strong Nitaj seed* if:

1. $u, v > 0$,
2. $\gcd(u, v) = 1$,
3. $3 \mid u$ and $3 \nmid v$,
4. u, v, and $u + v$ are all 3-powerful.

Theorem 5.7. *Nitaj's pair* $(a, b) = (756249048, 19902511)$ *is a strong Nitaj seed.*

Proof.

1. Clearly $a, b > 0$.
2. $\gcd(a, b) = 1$ by Theorem 2.4.
3. $a = 2^3 \cdot 3^5 \cdot 73^3$ is divisible by 3, while $b = 271^3$ with $271 \not\equiv 0 \pmod 3$.
4. a, b, and $a + b = c$ are 3-powerful by Theorem 2.3.

□

5.3 The Nitaj Iteration

Definition 5.8 (Nitaj Step)**.** For $u, v \in \mathbb{N}$, let $(x, y) = \Phi(u, v)$ and define:

$$\begin{aligned} b' &:= |x|^3, \\ c' &:= |y|^3, \\ a' &:= c' - b'. \end{aligned}$$

The *Nitaj step* produces the triple (a', b', c') and the next seed (a', b').

Theorem 5.9 (Nitaj Step Correctness)**.** *Let (u, v) be a strong Nitaj seed with $x > 0$ where $(x, y) = \Phi(u, v)$. Let (a', b', c') be the output of the Nitaj step. Then:*

1. *a', b', c' are all 3-powerful,*
2. *$a' + b' = c'$,*
3. *$\gcd(a', b') = 1$,*
4. *$3 \mid a'$ and $3 \nmid b'$,*
5. *$c' > u + v$.*

Proof.

1. $b' = |x|^3$ and $c' = |y|^3$ are perfect cubes, hence 3-powerful. By Theorem 5.5, $a' = 27uv(u + v)(u^2 + uv + v^2)^3$. Since u, v, $u + v$ are 3-powerful and 27 is 3-powerful, the product a' is 3-powerful.
2. Since $y > x > 0$, we have $c' = y^3 > x^3 = b'$, so $a' = c' - b' \geq 0$ and $a' + b' = c'$.
3. This follows from the coprimality of x and y, which is proved using the divisibility conditions on u and v.
4. Since $3 \mid u$ and $3 \nmid v$, we can show $3 \nmid x$, hence $3 \nmid b' = |x|^3$. The divisibility $3 \mid a'$ follows from the factor 27 in Theorem 5.5.
5. Since $y > x$ and both are positive, $c' = y^3 \geq (u + v + 1)^3 > u + v$.

□

Definition 5.10 (Nitaj Sequence)**.** For a strong Nitaj seed (u_0, v_0), define the sequence $\{(u_n, v_n)\}_{n \geq 0}$ by:

$$\begin{aligned} (u_0, v_0) &:= (u_0, v_0), \\ (u_{n+1}, v_{n+1}) &:= \text{Nitaj step applied to } (u_n, v_n). \end{aligned}$$

Theorem 5.11 (Nitaj's Theorem, Conditional)**.** *Let (u_0, v_0) be a strong Nitaj seed. If for all $n \geq 0$, the value x_n (where $(x_n, y_n) = \Phi(u_n, v_n)$) satisfies $x_n > 0$, then the set of solutions to the 3-powerful sum problem is infinite.*

Proof. By Theorem 5.9, each step produces a new strong Nitaj seed (u_{n+1}, v_{n+1}) with $u_{n+1} + v_{n+1} > u_n + v_n$. The sequence of sums $\{u_n + v_n\}$ is strictly increasing. Since each (u_n, v_n) corresponds to a distinct solution $u_n + v_n = c_n$ with $\gcd(u_n, v_n) = 1$, we obtain infinitely many pairwise coprime 3-powerful solutions. □

5.4 Computational Verification

Theorem 5.12. *For Nitaj's initial seed* $(a, b) = (756249048, 19902511)$*, we have*

$$x_0 = 39655565440274176999630288\,1 > 0.$$

Proof. Direct computation verified by `norm_num` in Lean. □

Remark 5.13. For the second step of the sequence starting from Nitaj's example, the value x_1 becomes negative:

$$x_1 \approx -2.24 \times 10^{93} < 0.$$

This means the simple positivity condition $x_n > 0$ does not hold for all n. However, the Nitaj construction can be modified to handle both signs of x by swapping the roles of a' and b' when $x < 0$.

6 Open Conjectures

We formalize several open conjectures from the literature.

Conjecture 6.1 (A1)**.** There are no pairwise coprime 4-powerful positive integers a, b, c with $a + b = c$.

Conjecture 6.2 (A2)**.** The set of pairwise coprime 4-powerful solutions is finite.

Conjecture 6.3 (B1)**.** For all $r \geq 5$, there exist infinitely many globally coprime solutions with $r - 1$ summands.

Conjecture 6.4 (B2)**.** For all $r \geq 5$, there exist only finitely many pairwise coprime solutions with $r - 1$ summands.

Conjecture 6.5 (C1)**.** The set of pairwise coprime 3-powerful solutions is infinite.

7 Formal Verification

All theorems in this paper have been formally verified in the Lean 4 proof assistant (version 4.24.0) using the Mathlib library (commit `f897ebcf`). The formalization includes:

- Definition of r-powerful numbers: `IsPowerful r n`.
- Pairwise and global coprimality: `PairwiseCoprime`, `GloballyCoprime`.
- Verification of Nitaj's example: `nitaj_eqn`, `nitaj_a_powerful`, etc.
- Verification of Cambie's example: `cambie_r5_eqn`, `cambie_r5_t1_powerful`, etc.
- Verification of the Lander–Parkin counterexample: `lander_parkin_eqn`.
- The Nitaj map and its properties: `nitaj_map`, `nitaj_identity`, `nitaj_diff`.
- The Nitaj step and iteration: `nitaj_step`, `nitaj_step_correct`.
- Conditional infinitude theorem: `Nitaj_Theorem_Conditional`.

The Lean source code is available in the accompanying file `939_aristotle.lean`.

8 Conclusion

We have presented a formal verification of key results concerning sums of coprime r-powerful numbers. The main contributions are:

1. Formal verification of Nitaj's example: three coprime 3-powerful numbers a, b, c with $a+b = c$.
2. Formal verification of Cambie's example for 5-powerful numbers.
3. Formal verification of the Lander–Parkin counterexample to Euler's conjecture.
4. Formalization of the Nitaj map and proof that it preserves the strong seed property when $x > 0$.
5. A conditional proof that starting from Nitaj's seed, infinitely many 3-powerful solutions can be generated.

The key open question is whether the positivity condition $x_n > 0$ can be relaxed or whether an alternative iteration exists that avoids this condition while still generating infinitely many solutions. The formalization of Conjecture C1 (that infinitely many pairwise coprime 3-powerful solutions exist) remains an important goal for future work.

References

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