Formal Framework for Optimal Market Making: HJB Equations with Alpha Signals and Adverse Selection

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January 13, 2026

Abstract

We present a formal mathematical framework for optimal market making strategies, encompassing three interconnected models: (1) the Market Making with Alpha Signals model incorporating predictive signals, (2) the Adverse Selection with Price Reading model, and (3) the classical Avellaneda-Stoikov model. For each model, we rigorously define the parameter spaces, derive the Hamilton-Jacobi-Bellman (HJB) equations, and establish the optimal control formulations. We also present the ansatz for the value function and the optimal quote adjustment formulas.

1 Introduction

Market making involves continuously providing liquidity by posting bid and ask quotes. The market maker faces several challenges including inventory risk, adverse selection, and the incorporation of predictive signals. This paper formalizes three models that address these challenges within a unified mathematical framework.

2 Market Making with Alpha Signals

2.1 Parameter Space

Definition 2.1 (Market Parameters)**.** The market making model with alpha signals is characterized by the following parameters:

$$\mathcal{P} = \left(\kappa, \xi, \eta^+, \eta^-, \sigma, \theta, \Upsilon_{\mathrm{MO}}, \psi, \phi, \lambda^+, \lambda^-, \Upsilon_{\mathrm{LO}}\right) \tag{1}$$

where:

- $\kappa \in \mathbb{R}$: mean-reversion speed of the alpha signal
- $\xi \in \mathbb{R}$: volatility of the alpha signal
- $\eta^+, \eta^- \in \mathbb{R}$: jump sizes in alpha upon trade execution
- $\sigma \in \mathbb{R}$: price tick size
- $\theta \in \mathbb{R}$: baseline intensity of price movements
- $\Upsilon_{\mathrm{MO}} \in \mathbb{R}$: market order execution cost
- $\psi \in \mathbb{R}$: running inventory penalty coefficient
- $\phi \in \mathbb{R}$: quadratic inventory penalty coefficient
- $\lambda^+, \lambda^- \in \mathbb{R}$: arrival intensities of buy and sell orders
- $\Upsilon_{\mathrm{LO}} \in \mathbb{R}$: limit order spread

2.2 The Quasi-Variational Inequality

Let $H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ denote the value function, where $H(t, x, S, \alpha, q)$ represents the expected utility at time t with cash x, mid-price S, alpha signal α, and inventory q.

Definition 2.2 (HJBQVI for Alpha Signal Model)**.** The Hamilton-Jacobi-Bellman Quasi-Variational Inequality (HJBQVI) is given by:

$$\max\Big\{\mathcal{A}, \mathcal{B}, \mathcal{C}\Big\} = 0 \tag{2}$$

where the continuation operator $\mathcal{A}$ is defined as:

$$\begin{aligned}
\mathcal{A} = {} & \frac{\partial H}{\partial t} + \big((\alpha)^+ + \theta\big)\big[H(t, x, S + \sigma, \alpha, q) - H(t, x, S, \alpha, q)\big] \\
& + \big((-\alpha)^+ + \theta\big)\big[H(t, x, S - \sigma, \alpha, q) - H(t, x, S, \alpha, q)\big] \\
& - \kappa\alpha \frac{\partial H}{\partial \alpha} + \frac{1}{2}\xi^2 \frac{\partial^2 H}{\partial \alpha^2} - \phi q^2 \\
& + \lambda^+ \max\Big\{H\big(t, x + S + \Upsilon_{\text{LO}}, S, \alpha + \eta^+, q - 1\big) - H(t, x, S, \alpha, q), \\
& \qquad\qquad H(t, x, S, \alpha + \eta^+, q) - H(t, x, S, \alpha, q)\Big\} \\
& + \lambda^- \max\Big\{H\big(t, x - S + \Upsilon_{\text{LO}}, S, \alpha - \eta^-, q + 1\big) - H(t, x, S, \alpha, q), \\
& \qquad\qquad H(t, x, S, \alpha - \eta^-, q) - H(t, x, S, \alpha, q)\Big\}
\end{aligned} \tag{3}$$

The intervention operators $\mathcal{B}$ and $\mathcal{C}$ are:

$$\mathcal{B} = H\big(t, x + S - \Upsilon_{\text{MO}}, S, \alpha, q - 1\big) - H(t, x, S, \alpha, q) \tag{4}$$

$$\mathcal{C} = H\big(t, x - S - \Upsilon_{\text{MO}}, S, \alpha, q + 1\big) - H(t, x, S, \alpha, q) \tag{5}$$

Here, $(\cdot)^+ = \max(\cdot, 0)$ denotes the positive part.

2.3 Ansatz for the Value Function

Proposition 2.1 (Separation of Variables)**.** The value function admits the following ansatz:

$$H(t, x, S, \alpha, q) = x + qS + \tilde{h}(t, \alpha, q) \tag{6}$$

where $\tilde{h} : \mathbb{R} \times \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ is the reduced value function that depends only on time, the alpha signal, and inventory.

3 Adverse Selection Model with Price Reading

3.1 Parameter Space

Definition 3.1 (Adverse Selection Parameters)**.** For a market with N regimes and K order sizes, the adverse selection model is characterized by:

$$\mathcal{P}_{\text{AS}} = \big(\rho, \gamma, \sigma, \tilde{\zeta}, \tilde{J}, w, \Delta, \beta, \Lambda^b, \Lambda^a\big) \tag{7}$$

where:

- $\rho \in \mathbb{R}$: discount rate
- $\gamma \in \mathbb{R}$: risk aversion coefficient

- $\sigma \in \mathbb{R}$: price volatility
- $\tilde{\zeta} : \{1, \dots, N\} \times \{1, \dots, K\} \to \mathbb{R}$: regime-size dependent signal
- $\tilde{J} : \{1, \dots, N\} \to \mathbb{R}$: regime-dependent cost function
- $w : \{1, \dots, N\} \times \{1, \dots, K\} \to \mathbb{R}$: weight function
- $\Delta : \{1, \dots, K\} \to \mathbb{R}$: order size for each tier
- $\beta : \{1, \dots, N\} \times \mathbb{Z} \times (\mathbb{R}^K \times \mathbb{R}^K) \to \mathbb{R}$: running payoff function
- $\Lambda^b, \Lambda^a : \{1, \dots, N\} \times \{1, \dots, K\} \times \mathbb{R} \to \mathbb{R}$: fill rate functions

3.2 The Hamiltonian

Definition 3.2 (Hamiltonian for Adverse Selection)**.** For regime $n \in \{1, \dots, N\}$, inventory $q \in \mathbb{Z}$, and finite difference values $p^b, p^a : \{1, \dots, K\} \to \mathbb{R}$, the Hamiltonian is:

$$\tilde{\mathcal{H}}_n(q, p^b, p^a) = \sup_{\delta^b, \delta^a \in \mathbb{R}^K} \left\{ \beta_n(q, \delta^b, \delta^a) - \sum_{k=1}^{K} \Delta_k \left[\Lambda^b_{n,k}(\delta^b_k) p^b_k + \Lambda^a_{n,k}(\delta^a_k) p^a_k \right] \right\} \tag{8}$$

3.3 Finite Difference Operators

Definition 3.3 (Finite Difference Operators)**.** For a function $\vartheta : \mathbb{R} \to \mathbb{R}$ and step size $\Delta > 0$:

$$D^+[\vartheta](q) = \frac{\vartheta(q + \Delta) - \vartheta(q)}{\Delta} \tag{9}$$

$$D^-[\vartheta](q) = \frac{\vartheta(q - \Delta) - \vartheta(q)}{\Delta} \tag{10}$$

For indexed step sizes Δ_k, $k \in \{1, \dots, K\}$:

$$D^+_k[\vartheta](q) = \frac{\vartheta(q + \Delta_k) - \vartheta(q)}{\Delta_k} \tag{11}$$

$$D^-_k[\vartheta](q) = \frac{\vartheta(q - \Delta_k) - \vartheta(q)}{\Delta_k} \tag{12}$$

3.4 HJB Equation for Adverse Selection

Definition 3.4 (Adverse Selection HJB Equation)**.** The value function $\vartheta : \mathbb{R} \to \mathbb{R}$ satisfies:

$$-\rho\vartheta(q) - \frac{1}{2}\gamma\sigma^2 q^2 + \sum_{n=1}^{N} \tilde{\mathcal{H}}_n\big(q, D^+_k[\vartheta](q), D^-_k[\vartheta](q)\big) = 0 \tag{13}$$

4 Optimal Quote Adjustment

4.1 Extended Parameter Space

Definition 4.1 (Differentiable Parameters)**.** For the perturbation analysis, we extend the parameter space to include:

$$\mathcal{P}_{\text{diff}} = \big(\rho, \gamma, \sigma, \tilde{\zeta}, \tilde{J}, w, \Delta, \beta, \Lambda^b, \Lambda^a, c\big) \tag{14}$$

where $\tilde{\zeta}, \tilde{J}$ are now differentiable functions of the order flow imbalance, and $c : \{1, \dots, N\} \times \{1, \dots, K\} \times \mathbb{R} \to \mathbb{R}$ is the cost function for quote adjustments.

4.2 Perturbation Approximation

Definition 4.2 (First-Order Perturbation)**.** The value function admits a perturbation expansion:

$$\vartheta_\varepsilon(q) = \vartheta_0(q) + \varepsilon f(q) + O(\varepsilon^2) \tag{15}$$

where ϑ_0 is the zeroth-order solution and f is the first-order correction.

4.3 Optimal Quote Adjustment Formula

Proposition 4.1 (Optimal Quote Adjustment)**.** For regime n, order size tier k, and inventory q, the optimal bid quote adjustment is:

$$\delta^{b,*}_{n,k}(q;\varepsilon) = \delta^{b,*}_{n,k}(q;0) + \varepsilon \cdot \frac{1}{c_{n,k}(\delta^{b,*}_{n,k})} \left(D^+_k[f](q) + \frac{\mathcal{N}_{n,k}(q)}{\mathcal{D}_{n,k}(q)} \right) \tag{16}$$

where:

$$\mathcal{N}_{n,k}(q) = q \cdot w_{n,k} \cdot \tilde{J}'_n(I_n(q)) + (q + \Delta_k) \cdot \frac{\partial}{\partial \delta} \left[\Lambda^b_{n,k}(\delta) \tilde{\zeta}_{n,k}(\delta) \right] \bigg|_{\delta = \delta^{b,*}_{n,k}} \tag{17}$$

$$\mathcal{D}_{n,k}(q) = \Delta_k \cdot \frac{\partial \Lambda^b_{n,k}}{\partial \delta} \bigg|_{\delta = \delta^{b,*}_{n,k}} \tag{18}$$

and the order flow imbalance is:

$$I_n(q) = \sum_{j=1}^{K} w_{n,j} \left(\delta^{a,*}_{n,j}(q) - \delta^{b,*}_{n,j}(q) \right) \tag{19}$$

5 The Avellaneda-Stoikov Model

5.1 Parameter Space

Definition 5.1 (Avellaneda-Stoikov Parameters)**.** The classical Avellaneda-Stoikov model is characterized by:

$$\mathcal{P}_{\text{AS}} = \left(\gamma, \sigma, \Delta, H^b, H^a \right) \tag{20}$$

where:

- $\gamma \in \mathbb{R}$: risk aversion coefficient
- $\sigma \in \mathbb{R}$: price volatility
- $\Delta \in \mathbb{R}$: order size
- $H^b, H^a : \mathbb{R} \to \mathbb{R}$: Hamiltonian functions for bid and ask sides

5.2 HJB Equation

Definition 5.2 (Avellaneda-Stoikov HJB Equation)**.** The value function $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$-\frac{\partial \theta}{\partial t}(t,q) + \frac{1}{2}\gamma\sigma^2 q^2 - H^b \left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta} \right) - H^a \left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta} \right) = 0 \tag{21}$$

Remark 5.1. The arguments of H^b and H^a represent the marginal values of selling and buying one unit, respectively. The Hamiltonian functions H^b and H^a typically take the form:

$$H^b(p) = \sup_{\delta^b \geq 0} \left\{ \Lambda^b(\delta^b)(\delta^b - p) \right\} \tag{22}$$

$$H^a(p) = \sup_{\delta^a \geq 0} \left\{ \Lambda^a(\delta^a)(\delta^a + p) \right\} \tag{23}$$

where Λ^b, Λ^a are the fill rate functions.

6 Conclusion

We have presented a comprehensive formal framework for optimal market making that unifies three important models in the literature. The Market Making with Alpha Signals model incorporates predictive information through the alpha signal process, the Adverse Selection model accounts for information asymmetry through the order flow, and the Avellaneda-Stoikov model provides the foundational HJB framework. The formal definitions and equations presented here have been verified through mechanized proof in the Lean theorem prover.

References

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