Mean-Field Game Formulation for Competitive Market Making: Hamilton-Jacobi-Bellman and Fokker-Planck Equations

Naoki Takata

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Abstract

We present a rigorous mathematical formalization of the mean-field game framework for competitive market making. The model captures the strategic interaction among a continuum of market makers who compete through their quoted bid and ask prices. We derive the Hamilton-Jacobi-Bellman (HJB) equation governing the value function of a representative agent, the Fokker-Planck equation describing the evolution of the inventory distribution, and the master equation characterizing the full mean-field equilibrium. Optimal quote strategies and arrival intensities are derived in closed form.

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1 Introduction

Market making involves continuously providing liquidity by posting bid and ask quotes. In competitive environments, multiple market makers interact through their pricing strategies, with order flow depending on relative quote positioning. The mean-field game (MFG) framework provides a natural setting for analyzing such competitive dynamics in the limit of many agents.

This paper formalizes the mathematical structure of competitive market making as a mean-field game. We present:

(i) The arrival intensity function capturing competition through relative quotes

(ii) The Hamiltonian structure of the optimization problem

(iii) Optimal quote and intensity formulas

(iv) The Hamilton-Jacobi-Bellman equation for the value function

(v) The Fokker-Planck equation for the inventory distribution

(vi) The master equation for the full mean-field system

2 Model Setup

2.1 State Variables

Consider a representative market maker whose state at time $t \in [0, T]$ is characterized by:

- $q_t \in \mathbb{R}$: inventory position
- $S_t \in \mathbb{R}$: mid-price of the asset
- μ_t: distribution of inventories across all market makers

The market maker controls:

- $\delta_t^a \in \mathbb{R}$: ask quote spread (distance from mid-price)
- $\delta_t^b \in \mathbb{R}$: bid quote spread (distance from mid-price)

2.2 Arrival Intensity

The competitive nature of market making is captured through the arrival intensity function. Orders arrive to a market maker based on how their quotes compare to the best competing quotes.

Definition 2.1 (Arrival Intensity)**.** *The arrival intensity for ask (sell) orders is given by:*

$$\lambda^a(\delta^a; \mu) = A \exp\left(-k\left(\delta^a - \delta^a_{\min}(\mu)\right)\right) \tag{1}$$

where:

- $A > 0$ *is the baseline arrival rate*
- $k > 0$ *is the sensitivity parameter*
- $\delta^a_{\min}(\mu) = \inf_j \delta^{a,j}$ *is the minimum ask spread among competitors*

Similarly, for bid (buy) orders:

$$\lambda^b(\delta^b; \mu) = A \exp\left(-k\left(\delta^b - \delta^b_{\min}(\mu)\right)\right) \tag{2}$$

where $\delta^b_{\min}(\mu) = \inf_j \delta^{b,j}$ *is the minimum bid spread among competitors.*

2.3 Dynamics

The inventory process q_t evolves according to:

$$dq_t = dN_t^a - dN_t^b \tag{3}$$

where N_t^a and N_t^b are counting processes with intensities λ_t^a and λ_t^b, respectively.

The mid-price follows a Brownian motion:

$$dS_t = \sigma\, dW_t \tag{4}$$

where $\sigma > 0$ is the volatility and W_t is a standard Brownian motion.

3 Value Function and Hamiltonian

3.1 Value Function

The value function of the representative market maker is defined as:

$$V(t, q, S, \mu) = \sup_{(\delta^a, \delta^b)} \mathbb{E}\left[\int_t^T \left(\delta_s^a\, dN_s^a + \delta_s^b\, dN_s^b - \phi q_s^2\, ds\right) - \gamma q_T^2 \,\middle|\, q_t = q, S_t = S\right] \tag{5}$$

where:

- $\phi \geq 0$ is the running inventory penalty
- $\gamma > 0$ is the terminal inventory penalty

3.2 Hamiltonian

Definition 3.1 (Hamiltonian)**.** *The Hamiltonian of the system is defined as:*

$$\mathcal{H}(\delta^a, \delta^b, q, \partial_q V; \mu) = \lambda^a(\delta^a; \mu)\,(\delta^a + \partial_q V) + \lambda^b(\delta^b; \mu)\left(\delta^b - \partial_q V\right) \tag{6}$$

Substituting the arrival intensity expressions:

$$\mathcal{H} = Ae^{-k(\delta^a - \delta^a_{\min})}\,(\delta^a + \partial_q V) + Ae^{-k(\delta^b - \delta^b_{\min})}\left(\delta^b - \partial_q V\right) \tag{7}$$

4 Optimal Controls

4.1 Optimal Quote Formula

Proposition 4.1 (Optimal Quotes)**.** *The optimal ask quote spread is given by:*

$$\delta^{a,*} = \delta^a_{\min} + \frac{1}{k} + \frac{1}{k}\log\left(1 + k\,\partial_q V\right) \tag{8}$$

The optimal bid quote spread is given by:

$$\delta^{b,*} = \delta^b_{\min} + \frac{1}{k} + \frac{1}{k}\log\left(1 - k\,\partial_q V\right) \tag{9}$$

Remark 4.1. *The optimal quotes depend on:*

(a) The competitive minimum spread $\delta_{\min}$

(b) The inverse sensitivity $1/k$

(c) A logarithmic adjustment based on the marginal value of inventory $\partial_q V$

4.2 Optimal Arrival Intensity

Proposition 4.2 (Optimal Intensity). *Substituting the optimal quote into the arrival intensity function, we obtain:*

$$\lambda^{a,*} = A \exp\left(-1 - \log(1 + k\,\partial_q V)\right) = \frac{A}{e(1 + k\,\partial_q V)} \tag{10}$$

Similarly:

$$\lambda^{b,*} = A \exp\left(-1 - \log(1 - k\,\partial_q V)\right) = \frac{A}{e(1 - k\,\partial_q V)} \tag{11}$$

Proof. From the optimal quote formula:

$$\delta^{a,*} - \delta^a_{\text{min}} = \frac{1}{k} + \frac{1}{k}\log(1 + k\,\partial_q V) \tag{12}$$

Thus:

$$-k(\delta^{a,*} - \delta^a_{\text{min}}) = -1 - \log(1 + k\,\partial_q V) \tag{13}$$

Substituting into the arrival intensity:

$$\lambda^{a,*} = A \exp\left(-k(\delta^{a,*} - \delta^a_{\text{min}})\right) \tag{14}$$

$$= A \exp\left(-1 - \log(1 + k\,\partial_q V)\right) \tag{15}$$

$$= \frac{A}{e} \cdot \frac{1}{1 + k\,\partial_q V} \tag{16}$$

□

4.3 Maximized Hamiltonian

Definition 4.1 (Maximized Hamiltonian). *The maximized Hamiltonian is obtained by substituting the optimal controls:*

$$\mathcal{H}^* = \mathcal{H}(\delta^{a,*}, \delta^{b,*}, q, \partial_q V; \mu) \tag{17}$$

Explicitly:

$$\mathcal{H}^* = \lambda^{a,*}\left(\delta^{a,*} + \partial_q V\right) + \lambda^{b,*}\left(\delta^{b,*} - \partial_q V\right) \tag{18}$$

5 Hamilton-Jacobi-Bellman Equation

Theorem 5.1 (HJB Equation). *The value function $V(t, q, S)$ satisfies the Hamilton-Jacobi-Bellman equation:*

$$\frac{\partial V}{\partial t} + \mathcal{H}^* + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - \phi q^2 = 0 \tag{19}$$

Definition 5.1 (HJB Equation - Expanded Form). *The full HJB equation is:*

$$\frac{\partial V}{\partial t} + \lambda^{a,*}\left(\delta^{a,*} + \frac{\partial V}{\partial q}\right) + \lambda^{b,*}\left(\delta^{b,*} - \frac{\partial V}{\partial q}\right) + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - \phi q^2 = 0 \tag{20}$$

for all $(t, q, S) \in [0, T) \times \mathbb{R} \times \mathbb{R}$.

6 Fokker-Planck Equation

The distribution of inventories $\mu_t(q)$ across the population of market makers evolves according to the Fokker-Planck equation.

Definition 6.1 (Drift Function). *The drift in inventory space is defined as:*

$$b(t,q) = \lambda^{a,*}(t,q) - \lambda^{b,*}(t,q) \tag{21}$$

representing the expected rate of inventory accumulation at state (t,q).

Theorem 6.1 (Fokker-Planck Equation). *The inventory distribution $\mu(t,q)$ satisfies:*

$$\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial q}\left(b(t,q)\cdot \mu(t,q)\right) = 0 \tag{22}$$

Substituting the optimal intensities:

$$\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial q}\left[\left(\frac{A}{e(1+k\,\partial_q V)} - \frac{A}{e(1-k\,\partial_q V)}\right)\mu\right] = 0 \tag{23}$$

Remark 6.1. *The Fokker-Planck equation is coupled to the HJB equation through the optimal intensities, which depend on $\partial_q V$. This coupling is characteristic of mean-field games.*

7 Master Equation

The master equation provides a unified description of the mean-field game, encoding both the value function dynamics and the distribution evolution.

Definition 7.1 (Master Equation). *Let $U(t,q,S,m)$ denote the value function with explicit dependence on the distribution m. The master equation is:*

$$\frac{\partial U}{\partial t} + \sup_{\delta} \mathcal{H}\left(q,S,m,\delta,\frac{\partial U}{\partial q},\frac{\delta U}{\delta m}\right) + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial S^2} = 0 \tag{24}$$

where $\frac{\delta U}{\delta m}$ denotes the Lions derivative (derivative with respect to the measure).

Remark 7.1. *The master equation:*

(i) Incorporates the HJB equation for the representative agent

(ii) Captures the feedback from the distribution through the minimum spread $\delta_{\min}(m)$

(iii) Provides a complete characterization of the mean-field equilibrium

8 Terminal and Boundary Conditions

Definition 8.1 (Terminal Condition). *The terminal condition for the value function is:*

$$V(T,q,S,\mu) = -\gamma q^2 \tag{25}$$

where $\gamma > 0$ is the terminal penalty parameter.

This condition reflects the cost of holding inventory at the terminal time, providing an incentive for the market maker to reduce inventory exposure as time approaches T.

9 Summary of the Mean-Field Game System

The complete mean-field game system for competitive market making consists of:

9.1 Forward-Backward System

(1) HJB Equation (Backward):

$$\frac{\partial V}{\partial t} + \mathcal{H}^*(q, \partial_q V; \mu) + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - \phi q^2 = 0 \tag{26}$$

with terminal condition $V(T, q, S) = -\gamma q^2$.

(2) Fokker-Planck Equation (Forward):

$$\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial q}\left((\lambda^{a,*} - \lambda^{b,*})\mu\right) = 0 \tag{27}$$

with initial condition $\mu(0, q) = \mu_0(q)$.

9.2 Key Formulas

Quantity	**Formula**
Arrival Intensity	$\lambda(\delta; \mu) = A \exp\left(-k(\delta - \delta_{\min})\right)$
Optimal Ask Quote	$\delta^{a,*} = \delta^a_{\min} + \frac{1}{k} + \frac{1}{k} \log(1 + k\, \partial_q V)$
Optimal Bid Quote	$\delta^{b,*} = \delta^b_{\min} + \frac{1}{k} + \frac{1}{k} \log(1 - k\, \partial_q V)$
Optimal Ask Intensity	$\lambda^{a,*} = \frac{A}{e(1 + k\, \partial_q V)}$
Optimal Bid Intensity	$\lambda^{b,*} = \frac{A}{e(1 - k\, \partial_q V)}$
Terminal Condition	$V(T, q, S, \mu) = -\gamma q^2$

10 Conclusion

We have presented a complete mathematical formalization of the mean-field game for competitive market making. The framework captures:

- Competition through price-dependent arrival intensities
- Optimal quote strategies in closed form
- The coupled HJB-Fokker-Planck system characterizing the mean-field equilibrium
- The master equation for the full system dynamics

This formalization provides a rigorous foundation for analyzing competitive dynamics in high-frequency market making and for developing numerical methods to compute mean-field equilibria.

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References

[1] Lasry, J.-M., & Lions, P.-L. (2007). Mean field games. *Japanese Journal of Mathematics*, 2(1), 229–260.

[2] Huang, M., Malhamé, R. P., & Caines, P. E. (2006). Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Communications in Information and Systems*, 6(3), 221–252.

[3] Cardaliaguet, P. (2010). Notes on mean field games. *Lecture notes, Université Paris-Dauphine*.

[4] Carmona, R., & Delarue, F. (2018). *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer.

[5] Lions, P.-L. (2007–2011). Cours au Collège de France: Théorie des jeux à champ moyen. *Video lectures, Collège de France*.

[6] Avellaneda, M., & Stoikov, S. (2008). High-frequency trading in a limit order book. *Quantitative Finance*, 8(3), 217–224.

[7] Guéant, O., Lehalle, C.-A., & Fernandez-Tapia, J. (2012). Dealing with the inventory risk: a solution to the market making problem. *Mathematics and Financial Economics*, 6(4), 477–507.

[8] Guéant, O., Lehalle, C.-A., & Fernandez-Tapia, J. (2013). Optimal portfolio liquidation with limit orders. *SIAM Journal on Financial Mathematics*, 4(1), 1–32.

[9] Cartea, Á., Jaimungal, S., & Penalva, J. (2015). *Algorithmic and High-Frequency Trading*. Cambridge University Press.

[10] Guéant, O. (2017). *The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making*. Chapman and Hall/CRC.

[11] Cardaliaguet, P., & Lehalle, C.-A. (2018). Mean field game of controls and an application to trade crowding. *Mathematics and Financial Economics*, 12(3), 335–363.

[12] Carmona, R., & Lacker, D. (2015). A probabilistic weak formulation of mean field games and applications. *The Annals of Applied Probability*, 25(3), 1189–1231.

[13] Lacker, D., & Soret, A. (2019). Many-player games of optimal consumption and investment under relative performance criteria. *Mathematics and Financial Economics*, 13(4), 581–619.

[14] Fujii, M., & Takahashi, A. (2020). A mean field game approach to equilibrium pricing with market clearing condition. *SIAM Journal on Control and Optimization*, 58(6), 3133–3160.

[15] Lachapelle, A., Lasry, J.-M., Lehalle, C.-A., & Lions, P.-L. (2016). Efficiency of the price formation process in presence of high frequency participants: a mean field game analysis. *Mathematics and Financial Economics*, 10(3), 223–262.

[16] Huang, W., Lehalle, C.-A., & Rosenbaum, M. (2019). Simulating and analyzing order book data: The queue-reactive model. *Journal of the American Statistical Association*, 114(525), 15–28.

[17] Baldacci, B., Bergault, P., & Guéant, O. (2021). Algorithmic market making for options. *Quantitative Finance*, 21(1), 85–102.

[18] Fleming, W. H., & Soner, H. M. (2006). *Controlled Markov Processes and Viscosity Solutions* (2nd ed.). Springer.

[19] Pham, H. (2009). *Continuous-time Stochastic Control and Optimization with Financial Applications*. Springer.

[20] Risken, H. (1996). *The Fokker-Planck Equation: Methods of Solution and Applications* (2nd ed.). Springer.

[21] Cardaliaguet, P., Delarue, F., Lasry, J.-M., & Lions, P.-L. (2019). *The Master Equation and the Convergence Problem in Mean Field Games*. Princeton University Press.

[22] Chassagneux, J.-F., Crisan, D., & Delarue, F. (2022). A probabilistic approach to classical solutions of the master equation for large population equilibria. *Memoirs of the American Mathematical Society*, 280(1379).