# Optimal Market Making with Hawkes Process Dynamics: An Impulse Control Formulation

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**Abstract**

We present a rigorous mathematical formulation of the optimal market making problem driven by multidimensional Hawkes processes under an impulse control framework. The model incorporates self-exciting order flow dynamics, inventory risk management, and strategic order placement. We derive the Hamilton-Jacobi-Bellman quasi-variational inequality (HJB-QVI) characterizing the value function and establish the terminal boundary condition for the optimization problem.

## 1 Introduction

Market making involves continuously providing liquidity by posting bid and ask quotes. The arrival of market orders exhibits clustering behavior, where trades tend to trigger subsequent trades. This phenomenon is naturally captured by Hawkes processes [**?**], which are self-exciting point processes.

The foundational work on optimal market making was established by Avellaneda and Stoikov [**?**], who derived closed-form solutions under diffusive price dynamics. Subsequent research has incorporated more realistic order flow models, including Hawkes processes [**?**, **?**]. The application of Hawkes processes to high-frequency finance has been extensively studied [**?**, **?**].

In this paper, we formulate the optimal market making problem as an impulse control problem [**?**, **?**]. The market maker chooses discrete intervention times to adjust their quotes, balancing the trade-off between capturing the bid-ask spread and managing inventory risk. This approach extends the continuous control framework to allow for discrete strategic interventions [**?**].

## 2 Multidimensional Hawkes Process

### 2.1 Parameters

Let $d \in \mathbb{N}$ denote the dimension of the Hawkes process. The process is characterized by the following parameters:

**Definition 2.1** (Hawkes Parameters)**.** The parameter set $\Theta = (\mu, \alpha, \gamma)$ consists of:

$$\mu : \{1, \dots, d\} \to \mathbb{R}_{\geq 0}, \quad \text{(baseline intensity)} \tag{1}$$

$$\alpha : \{1, \dots, d\} \times \{1, \dots, d\} \to \mathbb{R}_{\geq 0}, \quad \text{(excitation coefficients)} \tag{2}$$

$$\gamma : \{1, \dots, d\} \times \{1, \dots, d\} \to \mathbb{R}_{>0}. \quad \text{(decay rates)} \tag{3}$$

### 2.2 Intensity Function

**Definition 2.2** (Hawkes Intensity)**.** Let $\{T_k^{(j)}\}_{k\geq 1}$ denote the sequence of jump times for the $j$-th component. The intensity of the $i$-th component at time $t$ is defined as:

$$\lambda_i(t) = \mu_i + \sum_{j=1}^{d} \alpha_{ij} \sum_{T_k^{(j)}<t} \exp\left(-\gamma_{ij}(t - T_k^{(j)})\right). \tag{4}$$

**Theorem 2.1** (Explicit Representation)**.** The intensity function admits the equivalent representation:

$$\lambda_i(t) = \mu_i + \sum_{j=1}^{d} \alpha_{ij} \int_0^t e^{-\gamma_{ij}(t-s)}\, dN_j(s), \tag{5}$$

where $N_j(t) = \sum_{k\geq 1} \mathbf{1}_{\{T_k^{(j)}\leq t\}}$ is the counting process for the $j$-th component.

*Proof.* By the definition of the Lebesgue-Stieltjes integral with respect to the counting measure:

$$\int_0^t e^{-\gamma_{ij}(t-s)}\, dN_j(s) = \sum_{k:T_k^{(j)}\leq t} e^{-\gamma_{ij}(t-T_k^{(j)})} \tag{6}$$

$$= \sum_{T_k^{(j)}<t} e^{-\gamma_{ij}(t-T_k^{(j)})} + \mathbf{1}_{\{\exists k:T_k^{(j)}=t\}} \cdot 1. \tag{7}$$

Since the Hawkes process has no simultaneous jumps almost surely, and we consider the left-continuous version of the intensity (predictable), the two representations coincide. □

# 3 Market State Space

## 3.1 State Variables

**Definition 3.1** (Market State)**.** The market state $S \in \mathcal{S}$ is a tuple:

$$S = (X, Y, P_{\text{mid}}, \lambda, p, q, q_D, n), \tag{8}$$

where:

$$X \in \mathbb{R} \quad \text{(cash position)}, \tag{9}$$

$$Y \in \mathbb{R} \quad \text{(inventory position)}, \tag{10}$$

$$P_{\text{mid}} \in \mathbb{R}_{>0} \quad \text{(mid-price)}, \tag{11}$$

$$\lambda : \{1, \ldots, d\} \to \mathbb{R}_{\geq 0} \quad \text{(current intensities)}, \tag{12}$$

$$p : \{a, b\} \to \mathbb{R} \quad \text{(quoted prices)}, \tag{13}$$

$$q : \{a, b\} \to \mathbb{R}_{\geq 0} \quad \text{(quoted quantities)}, \tag{14}$$

$$q_D : \{a, b\} \to \mathbb{R}_{\geq 0} \quad \text{(market depths)}, \tag{15}$$

$$n : \{a, b\} \to \mathbb{R}_{\geq 0} \quad \text{(order counts)}. \tag{16}$$

Here, $a$ denotes the ask side and $b$ denotes the bid side.

# 4 Infinitesimal Generator

## 4.1 Generator Parameters

For computational tractability, we consider a simplified parameterization:

$$\Theta_L = (\mu, \alpha, \gamma) : \{1, \ldots, d\} \to \mathbb{R}^3, \tag{17}$$

where each component has its own baseline intensity $\mu_i$, self-excitation coefficient $\alpha_i$, and decay rate $\gamma_i$.

### 4.2 Transition Operators

Let $T_i : \mathcal{S} \to \mathcal{S}$ denote the transition operator when a jump occurs in the $i$-th component. After a jump, the intensity updates according to:

$$\lambda_j^+ = \begin{cases} \lambda_j + \alpha_i & \text{if } j = i, \\ \lambda_j & \text{otherwise.} \end{cases} \tag{18}$$

**Definition 4.1** (Infinitesimal Generator)**.** For a sufficiently smooth function $\Phi : [0, T] \times \mathcal{S} \to \mathbb{R}$, the infinitesimal generator $\mathcal{L}$ is defined as:

$$\mathcal{L}\Phi(t, S) = \frac{\partial \Phi}{\partial t}(t, S) + \mathcal{L}_{\text{jump}}\Phi(t, S) + \mathcal{L}_{\text{drift}}\Phi(t, S), \tag{19}$$

where the jump component is:

$$\mathcal{L}_{\text{jump}}\Phi(t, S) = \sum_{i=1}^{d} \lambda_i \left[\Phi\left(t, T_i(S)|_{\lambda_i \mapsto \lambda_i + \alpha_i}\right) - \Phi(t, S)\right], \tag{20}$$

and the drift component captures the mean-reversion of intensities:

$$\mathcal{L}_{\text{drift}}\Phi(t, S) = \sum_{i=1}^{d} \frac{\partial \Phi}{\partial \lambda_i}(t, S) \cdot \gamma_i(\mu_i - \lambda_i). \tag{21}$$

**Remark 4.1.** The drift term $\gamma_i(\mu_i - \lambda_i)$ arises from the Ornstein-Uhlenbeck-type dynamics of the intensity between jumps:

$$\frac{d\lambda_i}{dt} = -\gamma_i(\lambda_i - \mu_i), \quad t \in (T_k^{(i)}, T_{k+1}^{(i)}). \tag{22}$$

## 5 Impulse Control Formulation

### 5.1 Control Space

Let $\mathcal{U}$ denote the set of admissible controls. An impulse control strategy consists of:

- A sequence of intervention times $\{\tau_k\}_{k \geq 1}$ with $\tau_k < \tau_{k+1}$,
- A sequence of control actions $\{u_k\}_{k \geq 1}$ with $u_k \in \mathcal{U}$.

### 5.2 Intervention Operator

**Definition 5.1** (State Transition under Intervention)**.** Let $\Gamma : \mathcal{S} \times \mathcal{U} \to \mathcal{S}$ denote the state transition function under intervention. For a control action $u \in \mathcal{U}$:

$$S^+ = \Gamma(S, u). \tag{23}$$

**Definition 5.2** (Intervention Cost)**.** Let $K : \mathcal{S} \times \mathcal{U} \to \mathbb{R}$ denote the cost (or reward) associated with an intervention:

$$K(S, u) = (\text{immediate payoff from executing control } u \text{ in state } S). \tag{24}$$

**Definition 5.3** (Intervention Operator)**.** The intervention operator $\mathcal{M}$ acting on the value function $V$ is defined as:

$$\mathcal{M}V(t, S) = \sup_{u \in \mathcal{U}} \left\{V(t, \Gamma(S, u)) + K(S, u)\right\}. \tag{25}$$

# 6 Hamilton-Jacobi-Bellman Quasi-Variational Inequality

## 6.1 Value Function

The value function $V : [0, T] \times \mathcal{S} \to \mathbb{R}$ represents the optimal expected payoff starting from time $t$ in state $S$:

$$V(t, S) = \sup_{\pi \in \Pi} \mathbb{E} \left[ \int_t^T f(S_s)\, ds + g(S_T) \,\middle|\, S_t = S \right], \tag{26}$$

where $\Pi$ denotes the set of admissible impulse control strategies.

## 6.2 HJB Quasi-Variational Inequality

**Theorem 6.1** (HJB-QVI)**.** The value function $V$ satisfies the following quasi-variational inequality:

$$\min \{-\mathcal{L}V(t, S),\, V(t, S) - \mathcal{M}V(t, S)\} = 0, \tag{27}$$

for all $(t, S) \in [0, T) \times \mathcal{S}$.

**Remark 6.1.** The QVI has the following interpretation:
- **Continuation region**: $\{(t, S) : V(t, S) > \mathcal{M}V(t, S)\}$, where $-\mathcal{L}V = 0$.
- **Intervention region**: $\{(t, S) : V(t, S) = \mathcal{M}V(t, S)\}$, where immediate intervention is optimal.

# 7 Market Making Specification

## 7.1 Market Parameters

**Definition 7.1** (Market Parameters)**.** The market making problem is characterized by:

$$\eta \in \mathbb{R}_{\geq 0} \quad \text{(running inventory penalty coefficient)}, \tag{28}$$

$$\kappa \in \mathbb{R}_{\geq 0} \quad \text{(terminal inventory penalty coefficient)}. \tag{29}$$

## 7.2 Running Cost

**Definition 7.2** (Running Cost)**.** The running cost function penalizes inventory risk:

$$f(S) = -\eta \cdot Y^2, \tag{30}$$

where $Y$ is the inventory position.

**Remark 7.1.** The quadratic penalty $-\eta Y^2$ captures the market maker's aversion to holding large inventory positions, which expose them to adverse price movements.

## 7.3 Terminal Payoff

**Definition 7.3** (Terminal Payoff)**.** At the terminal time $T$, the market maker liquidates their position at the mid-price with a penalty:

$$g(S) = X + Y \cdot P_{\text{mid}} - \kappa \cdot Y^2. \tag{31}$$

**Remark 7.2.** The terminal payoff consists of:
- $X$: accumulated cash from trading,
- $Y \cdot P_{\text{mid}}$: mark-to-market value of inventory,
- $-\kappa Y^2$: liquidation cost penalty for non-zero terminal inventory.

### 7.4 Specific HJB-QVI

**Theorem 7.1** (Market Making HJB-QVI)**.** The value function for the optimal market making problem satisfies:

$$\min\{-\mathcal{L}V(t,S) - f(S),\, V(t,S) - \mathcal{M}V(t,S)\} = 0, \tag{32}$$

which expands to:

$$\min\left\{-\mathcal{L}V(t,S) + \eta Y^2,\, V(t,S) - \mathcal{M}V(t,S)\right\} = 0. \tag{33}$$

### 7.5 Terminal Condition

**Theorem 7.2** (Terminal Boundary Condition)**.** The value function satisfies the terminal condition:

$$V(T,S) = g(S) = X + Y \cdot P_{\text{mid}} - \kappa \cdot Y^2, \tag{34}$$

for all $S \in \mathcal{S}$.

## 8 Summary

We have presented a complete mathematical formulation of the optimal market making problem with Hawkes process dynamics under an impulse control framework. The key components are:

1. **Hawkes Process Dynamics**: The intensity function

$$\lambda_i(t) = \mu_i + \sum_{j=1}^{d} \alpha_{ij} \sum_{T_k^{(j)} < t} e^{-\gamma_{ij}(t - T_k^{(j)})}. \tag{35}$$

2. **Infinitesimal Generator**:

$$\mathcal{L}\Phi = \frac{\partial \Phi}{\partial t} + \sum_{i=1}^{d} \lambda_i \left[\Phi(T_i(S)^+) - \Phi(S)\right] + \sum_{i=1}^{d} \frac{\partial \Phi}{\partial \lambda_i} \gamma_i(\mu_i - \lambda_i). \tag{36}$$

3. **Intervention Operator**:

$$\mathcal{M}V(t,S) = \sup_{u \in \mathcal{U}} \{V(t, \Gamma(S,u)) + K(S,u)\}. \tag{37}$$

4. **HJB-QVI**:

$$\min\left\{-\mathcal{L}V + \eta Y^2,\, V - \mathcal{M}V\right\} = 0. \tag{38}$$

5. **Terminal Condition**:

$$V(T,S) = X + Y \cdot P_{\text{mid}} - \kappa Y^2. \tag{39}$$

This formulation provides the foundation for numerical solution methods and further analytical study of optimal market making strategies under self-exciting order flow dynamics.

## References


[1] F. Abergel and A. Jedidi. A mathematical approach to order book modeling. *International Journal of Theoretical and Applied Finance*, 16(5):1350025, 2013.

[2] M. Avellaneda and S. Stoikov. High-frequency trading in a limit order book. *Quantitative Finance*, 8(3):217–224, 2008.

[3] E. Bacry, I. Mastromatteo, and J.-F. Muzy. Hawkes processes in finance. *Market Microstructure and Liquidity*, 1(1):1550005, 2015.

[4] A. Bensoussan and J.-L. Lions. *Impulse Control and Quasi-Variational Inequalities.* Gauthier-Villars, Paris, 1984.

[5] P. Cartea, S. Jaimungal, and J. Penalva. *Algorithmic and High-Frequency Trading.* Cambridge University Press, 2015.

[6] O. Guéant, C.-A. Lehalle, and J. Fernandez-Tapia. Optimal portfolio liquidation with limit orders. *SIAM Journal on Financial Mathematics*, 3(1):498–539, 2012.

[7] A. G. Hawkes. Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58(1):83–90, 1971.

[8] U. Horst and W. Xu. A scaling limit for limit order books driven by Hawkes processes. *SIAM Journal on Financial Mathematics*, 10(2):350–393, 2019.

[9] B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions.* Springer, Berlin, 2nd edition, 2007.

[10] X. Lu and F. Abergel. High-dimensional Hawkes processes for limit order books: Modelling, empirical analysis and numerical calibration. *Quantitative Finance*, 18(2):249–264, 2018.

[11] T. Jaisson and M. Rosenbaum. Limit theorems for nearly unstable Hawkes processes. *The Annals of Applied Probability*, 25(2):600–631, 2015.

[12] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions.* Springer, New York, 2nd edition, 2006.