On the Falsity of Certain Upper Bounds for Clique Transversal Numbers

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Abstract

We investigate proposed upper bounds for the clique transversal number $\tau(G)$ of a graph G on n vertices. Specifically, we disprove two conjectured bounds: $\tau(G) \leq n - \omega(n)\sqrt{n}$ for some function $\omega(n) \to \infty$, and $\tau(G) \leq n - c\sqrt{n \log n}$ for some constant $c > 0$. The counterexample in both cases is the empty graph, for which $\tau(G) = n$. All results have been formally verified in the Lean 4 theorem prover using the Mathlib library.

1 Introduction

Let $G = (V, E)$ be a simple graph. A *clique* in G is a subset of vertices that are pairwise adjacent. A clique is *maximal* if it is not properly contained in any other clique. A *clique transversal* is a set of vertices that intersects every maximal clique of G.

The study of clique transversal numbers has applications in various areas including network design and combinatorial optimization. In this paper, we show that certain natural upper bounds on the clique transversal number are false.

2 Preliminaries

Definition 2.1 (Clique Transversal)**.** Let $G = (V, E)$ be a graph with $|V| = n$. A set $S \subseteq V$ is called a *clique transversal* of G if for every maximal clique C of G, we have $S \cap C \neq \emptyset$.

Definition 2.2 (Clique Transversal Number)**.** The *clique transversal number* of G, denoted $\tau(G)$, is the minimum cardinality of a clique transversal of G:

$$\tau(G) = \min\{|S| : S \text{ is a clique transversal of } G\}.$$

3 The Empty Graph

The key observation for our counterexamples is the following lemma about the empty graph.

Lemma 3.1. *Let K_n^c denote the empty graph on n vertices (i.e., the graph with no edges). Then*

$$\tau(K_n^c) = n.$$

Proof. In the empty graph K_n^c, every vertex forms a maximal clique by itself (since there are no edges, no clique can have more than one vertex, and every singleton is maximal).

Thus, the maximal cliques of K_n^c are precisely the singletons $\{v\}$ for each $v \in V$. A clique transversal must intersect each of these n singletons, which means it must contain every vertex. Therefore, the minimum clique transversal has size exactly n. □

4 Main Results

We now show that two natural upper bounds for the clique transversal number are false.

Theorem 4.1. *There does not exist a function $\omega : \mathbb{N} \to \mathbb{R}$ with $\omega(n) \to \infty$ as $n \to \infty$ such that for all $n \in \mathbb{N}$ and all graphs G on n vertices,*

$$\tau(G) \leq n - \omega(n)\sqrt{n}.$$

Proof. Suppose for contradiction that such a function ω exists with $\omega(n) \to \infty$.

Since $\omega(n) \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\omega(n) > 1$.

Consider the empty graph $G = K_{N+1}^c$ on $N + 1$ vertices. By Lemma **??**,

$$\tau(G) = N + 1.$$

If the proposed bound held, we would have

$$N + 1 = \tau(G) \leq (N + 1) - \omega(N + 1)\sqrt{N + 1}.$$

This implies

$$\omega(N + 1)\sqrt{N + 1} \leq 0.$$

However, since $\omega(N + 1) > 1 > 0$ and $\sqrt{N + 1} > 0$, we have $\omega(N + 1)\sqrt{N + 1} > 0$, a contradiction. □

Theorem 4.2. *There does not exist a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all graphs G on n vertices,*

$$\tau(G) \leq n - c\sqrt{n \log n}.$$

Proof. Suppose for contradiction that such a constant $c > 0$ exists.

Consider the empty graph $G = K_2^c$ on 2 vertices. By Lemma **??**,

$$\tau(G) = 2.$$

If the proposed bound held, we would have

$$2 = \tau(G) \leq 2 - c\sqrt{2 \log 2}.$$

This implies

$$c\sqrt{2 \log 2} \leq 0.$$

However, since $c > 0$ and $\sqrt{2 \log 2} > 0$ (as $\log 2 > 0$), we have $c\sqrt{2 \log 2} > 0$, a contradiction. □

5 Formalization

All definitions and theorems presented in this paper have been formally verified using the Lean 4 theorem prover (version 4.24.0) with the Mathlib library (commit `f897ebcf72cd16f89ab4577d0c826cd14afaaf`

The formalization defines:

- `SimpleGraph.IsCliqueTransversal`: The predicate for a set being a clique transversal.
- `SimpleGraph.cliqueTransversalNumber`: The clique transversal number as the infimum of sizes of clique transversals.

The key theorems are:

- `empty_graph_cliqueTransversalNumber_eq_card`: Corresponds to Lemma **??**.
- `cliqueTransversal_bound_omega_false`: Corresponds to Theorem **??**.
- `cliqueTransversal_bound_const_false`: Corresponds to Theorem **??**.

6 Conclusion

We have shown that certain natural upper bounds on the clique transversal number involving sublinear corrections of the form $n - f(n)$ are false when $f(n)$ grows too quickly. The empty graph serves as a universal counterexample, as its clique transversal number achieves the maximum possible value of n.

This suggests that any valid upper bound on $\tau(G)$ that is strictly less than n must either:

1. Impose additional conditions on the graph G (e.g., minimum degree, connectivity, or edge density), or
2. Use a correction term that vanishes for sparse graphs.

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The formal verification was performed using Aristotle, an automated theorem proving system for Lean 4.