On the Minimal Size of Additive Bases for Initial Segments of Natural Numbers

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Abstract

We study the function $g(n)$, defined as the minimal cardinality of a subset $A \subseteq \{0, 1, \dots, n\}$ such that every element of $\{0, 1, \dots, n\}$ can be expressed as the sum of two elements of A. We establish an upper bound $g(n) \leq 2\sqrt{n}+2$ using a construction based on the Mrose–Rohrbach method, and a lower bound $g(n) \geq \sqrt{2n}-1$ via a counting argument. These bounds confirm that $g(n) \sim 2\sqrt{n}$ asymptotically. All results have been formally verified in the Lean 4 proof assistant using the Mathlib library.

1 Introduction

The problem of finding minimal additive bases has a rich history in additive combinatorics. Given a positive integer n, we seek the smallest subset A of $\{0, 1, \dots, n\}$ such that every element in this range can be represented as a sum of two elements from A. This is known as the *postage stamp problem* or the *Rohrbach problem* for bases of order 2.

Definition 1.1. For a finite set $A \subseteq \mathbb{N}$, we define the *sumset* $A + A$ as

$$A + A := \{a + b : a, b \in A\}.$$

We say that A is an *additive basis of order 2* for $\{0, 1, \dots, n\}$ if $\{0, 1, \dots, n\} \subseteq A + A$.

Definition 1.2. We define the function $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) := \min\{|A| : A \subseteq \{0, 1, \dots, n\} \text{ and } \{0, 1, \dots, n\} \subseteq A + A\}.$$

The main results of this paper are the following bounds:

Theorem 1.3 (Upper Bound). *For all $n \in \mathbb{N}$, we have*

$$g(n) \leq 2\sqrt{n} + 2.$$

Theorem 1.4 (Lower Bound). *For all $n \in \mathbb{N}$, we have*

$$g(n) \geq \sqrt{2n} - 1.$$

Corollary 1.5. *The function $g(n)$ satisfies $g(n) \sim 2\sqrt{n}$ as $n \to \infty$.*

2 Upper Bound: The Mrose–Rohrbach Construction

To prove the upper bound, we construct an explicit additive basis. The construction is based on the classical work of Rohrbach [1] and Mrose [2].

Definition 2.1 (Candidate Set)**.** For $n \geq 1$, let $m := \lfloor \sqrt{n} \rfloor$ and $K := \lceil n/m \rceil - 1$. We define the *candidate set* as

$$A_n := \{0, 1, \ldots, m\} \cup \{0, m, 2m, \ldots, Km\}.$$

For $n = 0$, we set $A_0 := \{0\}$.

The candidate set consists of two parts:

- A "small" set $B := \{0, 1, \ldots, m\}$ containing all residues modulo m.
- A "large" set $C := \{0, m, 2m, \ldots, Km\}$ containing multiples of m up to approximately n.

Lemma 2.2. *For all $n \in \mathbb{N}$, we have $A_n \subseteq \{0, 1, \ldots, n\}$.*

Proof. For $n = 0$, this is trivial. For $n \geq 1$, elements of $\{0, 1, \ldots, m\}$ are at most $\sqrt{n} \leq n$. For elements km with $k \leq K$, we have

$$km \leq Km = \left(\left\lceil \frac{n}{m} \right\rceil - 1 \right) \cdot m \leq \frac{n + m - 1}{m} \cdot m - m = n + m - 1 - m < n + 1.$$

Thus $km \leq n$ for all $k \leq K$. □

Lemma 2.3. *For all $n \in \mathbb{N}$, the set A_n is an additive basis of order 2 for $\{0, 1, \ldots, n\}$.*

Proof. Let $x \in \{0, 1, \ldots, n\}$. Write $x = qm + r$ where $q = \lfloor x/m \rfloor$ and $r = x \mod m$.

If $q \leq K$, then $qm \in C$ and $r \in B$, so $x = qm + r \in A_n + A_n$.

If $q > K$, we show that $x = Km + m$. Since $q > K$ and $x \leq n$, we have

$$(K + 1)m \leq qm \leq x \leq n.$$

But $K = \lceil n/m \rceil - 1$ implies $(K + 1)m \geq n$. Combined with $x \leq n$, we get $x = (K + 1)m = Km + m$. Since $Km \in C$ and $m \in B$, we have $x \in A_n + A_n$. □

Lemma 2.4. *For all $n \in \mathbb{N}$, we have $|A_n| \leq 2\sqrt{n} + 2$.*

Proof. For $n = 0$, we have $|A_0| = 1 \leq 2$.

For $n \geq 1$, note that $|B| = m + 1$ and $|C| = K + 1$. The sets B and C overlap at $\{0\}$, so by inclusion-exclusion:

$$|A_n| = |B| + |C| - |B \cap C| \leq (m + 1) + (K + 1) - 1 = m + K + 1.$$

Since $m = \lfloor \sqrt{n} \rfloor \leq \sqrt{n}$ and

$$K = \left\lceil \frac{n}{m} \right\rceil - 1 \leq \frac{n + m - 1}{m} - 1 = \frac{n - 1}{m} \leq \frac{n}{\sqrt{n}} = \sqrt{n},$$

we obtain

$$|A_n| \leq \sqrt{n} + \sqrt{n} + 1 = 2\sqrt{n} + 1 < 2\sqrt{n} + 2.$$

□

Proof of Theorem 1.3. By Lemmas 2.2, 2.3, and 2.4, the set A_n is an additive basis for $\{0, 1, \ldots, n\}$ with $|A_n| \leq 2\sqrt{n} + 2$. Therefore $g(n) \leq |A_n| \leq 2\sqrt{n} + 2$. □

3 Lower Bound: A Counting Argument

The lower bound follows from a simple counting argument based on the size of sumsets.

Lemma 3.1. *For any finite set $A \subseteq \mathbb{N}$ with $|A| = k$, we have*

$$|A + A| \leq \frac{k(k+1)}{2}.$$

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$. Consider the sums $a_i + a_j$ with $i \leq j$. These are pairwise distinct because if $a_i + a_j = a_{i'} + a_{j'}$ with $i \leq j$ and $i' \leq j'$, then by the strict monotonicity of the sequence, we must have $(i, j) = (i', j')$.

The number of pairs (i, j) with $1 \leq i \leq j \leq k$ is $\binom{k+1}{2} = \frac{k(k+1)}{2}$.

Since every element of $A+A$ can be written as a_i+a_j for some $i \leq j$, and these representations are injective (by the strict ordering), we have $|A + A| \leq \frac{k(k+1)}{2}$. □

Proof of Theorem 1.4. Suppose A is an additive basis of order 2 for $\{0, 1, \ldots, n\}$ with $|A| = k$. Then:

$$n + 1 = |\{0, 1, \ldots, n\}| \leq |A + A| \leq \frac{k(k+1)}{2}.$$

Thus $k(k+1) \geq 2(n+1) > 2n$, which implies $k^2+k > 2n$, so $k > \sqrt{2n + 1/4} - 1/2 > \sqrt{2n} - 1$.

Since k is an integer and $k > \sqrt{2n} - 1$, we have $g(n) = k \geq \lceil \sqrt{2n} - 1 \rceil$, and in particular $g(n) \geq \sqrt{2n} - 1$. □

4 Computational Results

We have computed $g(n)$ for small values of n by exhaustive search. The results are shown in Table 1.

n	0	1	2	3	4	5	6	7	8	9
$g(n)$	1	2	2	3	3	4	4	4	4	5

n	10	11	12	13	14	15	16	17	18	19
$g(n)$	5	5	5	6	6	6	6	7	7	7

Table 1: Computed values of $g(n)$ for $n = 0, \ldots, 19$.

We have also verified specific upper bounds:

- $g(30) \leq 10$, achieved by the set $A = \{0, 1, 2, 3, 7, 11, 15, 19, 23, 27\}$.
- $g(42) \leq 12$, achieved by the candidate set construction.

5 The Mrose Construction

For larger values, the Mrose construction provides an efficient additive basis. Define:

$$\begin{aligned}
B &:= \{0, 1, \ldots, a\}, \\
C &:= \{0, d, 2d, \ldots, \ell d\} \quad \text{where } d = a + 1, \\
D &:= \{0, e, 2e, \ldots, me\} \quad \text{where } e = \ell d + a + 1.
\end{aligned}$$

The Mrose set is $A := B \cup C \cup D$. With appropriate choices of parameters (e.g., $\ell = a$ and $m = 2a^2$), this construction achieves:

$$|A| \leq 2a^2 + 2a + 3.$$

This provides additive bases for ranges up to approximately $2a^4$, giving an asymptotic improvement for constructing efficient bases.

6 Formal Verification

All theorems in this paper have been formally verified in the Lean 4 proof assistant (version 4.24.0) using the Mathlib library (commit `f897ebcf`). The formalization includes:

- Definitions of additive bases and the function $g(n)$.
- Proofs of the upper bound $g(n) \leq 2\sqrt{n} + 2$.
- Proofs of the lower bound $g(n) \geq \sqrt{2n} - 1$.
- Verified computations for specific values.

The Lean source code is available in the accompanying file `791_aristotle.lean`.

7 Conclusion

We have established that the minimal size $g(n)$ of an additive basis of order 2 for $\{0, 1, \ldots, n\}$ satisfies

$$\sqrt{2n} - 1 \leq g(n) \leq 2\sqrt{n} + 2.$$

This confirms the asymptotic behavior $g(n) \sim 2\sqrt{n}$. The lower bound constant $\sqrt{2} \approx 1.414$ and the upper bound constant 2 bracket the true asymptotic constant. Determining the exact asymptotic constant remains an open problem, though it is generally believed to be 2.

References

[1] H. Rohrbach, *Ein Beitrag zur additiven Zahlentheorie*, Mathematische Zeitschrift, 42(1):1–30, 1937.

[2] A. Mrose, *Untere Schranken für die Reichweiten von Extremalbasen fester Ordnung*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 48:152–133, 1979.

[3] R. G. Stanton, *The postage stamp problem: An introduction for undergraduates*, Mathematics Magazine, 82(5):338–346, 2009.